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SOLVING HIGHER ORDER SINGULAR INITIAL VALUE PROBLEMS BY ADJUSTED ADOMIAN DECOMPOSITION STRATEGY

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Abstract: In this search, we assume the singular initial value problems of order (n+2). We give the two new operator for studying this problems and we give illustrations this method by some examples, the linear and nonlinear examples prove that the presented method is reliable, efficient, easy to implement

Keywords: Adomian decomposition strategy, singular initial value problem, higher order ordinary differential equation

1. Introduction

Assume the singular initial value problem of $(n+2)$ order as:

$$y^{(n+2)} + \frac{m}{x} y^{(n+1)} = g(x) + f(x,y) \quad , \ n \geq 1 \, , \ m > 0 \tag{1}$$

$$y(0) = y'(0) = y''(0) = \cdots = y^{(n+1)}(0) = 0$$

Where $g(x)$ *and* $f(x,y)$ is a real functions.

the singular initial value problems of ordinary differential equations which called Emdenfowler equations[5-8,12,13] assumed by many mathematicians and physicists, The well-known Emden-fowler equations have been used to model several phenomena in mathematical physics Adomian's decomposition method [1-3] presented by George Adomian, it is powerful and reliable method for solving various kinds of problems arising in applied sciences,, The method gives approximate solutions which converge rapidly to accurate solutions. Some modifications on the ADM was introduced by numerous different creators [4,6-9,12,13].

In this paper we introduce a new reliable modification of ADM a two new differential operator is defined which can be used for higher order singular initial value problems, the first for odd order (2n+1) and the second for even order (2(n+1)). Some numerical examples, with specified initial conditions will be examined to handle the singular point that exist in each equation.

2. Adomian Decomposition strategy (The First Adjusted) The first differential operator L is defined by:

$$L(.) = x^{-m} \frac{d}{dx} x^{m-n} \frac{d^n}{dx^n} x^{2n} \frac{d^n}{dx^n} x^{-n} (.) \tag{2}$$

Which gives the left side of differential equation as:

$$y^{(2n+1)} + \frac{m}{x} y^{(2n)} = g(x) + f(x,y) \quad , \ n \geq 1 \, , \ m > 0 \tag{3}$$

$$y(0) = y'(0) = y''(0) = \ \ldots = y^{(2n)}(0) = 0$$

Where $g(x)$ and $f(x,y)$ is a real functions.

We rewrite (3) in the form

$$Ly = g(x) + f(x,y) \tag{4}$$

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The inverse operator L^{-1} is defined, as below

$$L^{-1}(.) = x^n \int_0^x \dots \int_0^x x^{-2n} \int_0^x \dots \int_0^x x^{n-m} \int_0^x x^m\,(.)dx \dots dx \qquad (5)$$

Applying L^{-1} on (4) we find

$$L^{-1}(Ly) = L^{-1}(g(x)) + L^{-1}(f(x,y))$$

$$y(x) = L^{-1}(g(x)) + L^{-1}(f(x,y)) \qquad (6)$$

The Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (7)$$

and

$$f(x,y) = \sum_{n=0}^{\infty} A_n \qquad (8)$$

Where the $y_n(x)$ components of the solution will be determined recurrently. Specific algorithms were seen in [10-11] formulate Adomian polynomials. The following algorithm:

$$A_0 = f\ u_0 \qquad (\ )$$

$$A_1 = f'(u_0)u_1$$

$$A_2 = f'(u_0)u_2 + \frac{1}{2}f''(u_0)u_1^2$$

$$A_3 = f'(u_0)u_3 + f'(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0)u_1^3, \qquad (9)$$

$$\vdots$$

can be used to construct Adomian polynomials, when $f(u)$ is a nonlinear function.

By substituting (7) and (8) into (6),

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(g(x)) + L^{-1}\sum_{n=0}^{\infty} A_n \qquad (10)$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = L^{-1}(g(x))$$

$$y_{n+1}(x) = L^{-1}A_n \quad n \geq 0, \qquad (11)$$

Which gives

$$y_0 = L^{-1}(g(x)),$$

$$y_1 = L^{-1}A_0\,,$$

$$y_2 = L^{-1}A_1\,,$$

$$y_3 = L^{-1}A_2\,, \qquad (12)$$

$$\vdots$$

From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (7) can be immediately obtained.

3. Illustrative Examples

Example 1. We assume the non-linear initial value problem:

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$$y''' + \frac{}{x}y'' = 12 + x^6 - y^2, \quad (13)$$

$(0) = 0$, $'(0) = 0$, $''(0) = 0.$

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with exact solution $= x^3$. Eq.(13) can be written as

$$Ly = 12 + x^6 - y^2, \quad (14)$$

Where differential operator

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^2\frac{d}{dx}x^{-1}(.),$$

And inverse operator

$$L^{-1}(.) = x^1\int_0^x x^{-2}\int_0^x\int_0^x x\,(.)dxdxdx.$$

on both sides of (14), and using the initial conditions at x = 0, yields

$$y(x) = L^{-1}(12 + x^6) - L^{-1}(y^2), \quad (15)$$

Substituting the decomposition series for y(x) into (15y_n(x)) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(12 + x^6) - L^{-1}(y^2), \quad (16)$$

$$y_0 = L^{-1}(12 + x^6),$$

$$y_{n+1} = -L^{-1}(A_n)\,, n \geq 0. \quad (17)$$

$$A_0 = y_0^2\,, A_1 = 2y_0y_1,$$

$$A_2 = y_1^2 + 2y_0y_2\,, \quad (18)$$

...

Using (18), the first several calculated solution components are

$$y_0 = x^3 + \frac{1}{576}x^9\,,$$

$$y_1 = -L^{-1}(y_0^2) = -\frac{1}{576}x^9 - \frac{1}{846720}x^{15} - \frac{1}{2786918400}x^{21},$$

$$y_2 = -L^{-1}(2y_0y_1) = \frac{1}{846720}x^{15} + \frac{341}{341397504000}x^{21} + \frac{47}{178033921228800}x^{27} + \frac{1}{54245114825932800}x^{33}\,,$$

$$y_3 = -L^{-1}(y_1^2 + 2y_0y_2) = -\frac{437}{682795008000}x^{21} - \cdots$$

We note that:

$$\frac{1}{576}x^9 - \frac{1}{576}x^9 = 0$$

$$-\frac{1}{846720}x^{15} + \frac{1}{846720}x^{15} = 0$$

$$-\frac{1}{2786918400}x^{21} + \frac{341}{341397504000}x^{21} - \frac{437}{682795008000}x^{21} = 0$$

Other components can be evaluated in a similar manner. Which gives the exact solution

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$$y(x) = x^3 \quad (19)$$

Example 2. We assume the non-linear initial value problem:

$$y''' + \frac{2}{x}y'' + (18 + 36x^3)e^{-3y} = 0 \quad (20)$$

$$y(0) = 0\ ,\ \ y'(0) = 0\ ,\qquad y''(0) = 0$$

with exact solution = $ln(1 - x^3)$ Eq.(20) can be written as

$$Ly = -(18 + 36x^3)e^{-3y} \quad (21)$$

Where differential operator

$$L(.) = x^{-2}\frac{d}{dx}x\frac{d}{dx}x^2\frac{d}{dx}x^{-1}(.)$$

And inverse operator

$$L^{-1}(.) = x\int_0^x x^{-2}\int_0^x x^{-1}\int_0^x x^2\,(.)dxdxdx$$

on both sides of (21), and using the initial conditions at x = 0, yields

$$y(x) = L^{-1}((-18 - 36x^3)e^{-3y}) \quad (22)$$

Substituting the decomposition series $y_n(x)$ for y(x) into (22) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}((-18 - 36x^3)e^{-3y})\ , \quad (23)$$

$$y_0 = 0$$

$$y_{n+1} = L^{-1}((-18 - 36x^3)A_n), \qquad n \geq 0 \quad (24)$$

$$A_0 = e^{-3y_0}\ ,$$

$$A_1 = -3e^{-^3y_0}y_1,$$

$$A_2 = -3e^{-3y_0}y_2 + \frac{9}{2}e^{-3y_0}y_1^2\ , \quad (25)$$

...

$$y_0 = 0$$

$$y_1 = x^3 + \frac{1}{28}x^6 + \frac{1}{165}x^9$$

$$y_2 = -\frac{15}{28}x^6 - \frac{3}{70}x^9 - \frac{523}{56056}x^{12} - \frac{13}{61600}x^{15} - \frac{1}{62700}x^{18}$$

$$y_3 = -\frac{3}{20}x^9 - \frac{263}{1760}x^{12} - \frac{5413}{192500}x^{15} - \frac{27}{18700}x^{18}$$

$$y(x) = -x^3 - \frac{1}{2}x^6 - \frac{1}{3}x^9 - \ \ldots \quad (26)$$

That converges to the exact solution = $ln(1 - x^3)$ by Taylor series.

Example 3. We assume the linear initial value problem:

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$$y^{(5)}+\frac{}{x}y^{(4)}=4+x^5-5!\,y\,, \qquad (27)$$

$y(0)=0\,,\ y'(0)=0\,,\quad y''(0)=0\ ,\quad y'''(0)=0\ ,\quad y^{(4)}(0)=0\,.$

With exact solution $=\frac{1}{5!}x^5$.

Eq.(27) can be written as

$$Ly=4+x^5-5!\,y, \qquad (28)$$

Where differential operator

$$L(.)=x^{-3}\frac{d}{dx}x\frac{d^2}{dx^2}x^4\frac{d^2}{dx^2}x^{-2}(.),$$

And inverse operator

$$L^{-1}(.)=x^2\int_0^x\int_0^x x^{-4}\int_0^x\int_0^x x^{-1}\int_0^x x^3\,(.)dxdxdxdxdx.$$

On both sides of (28), and using the initial conditions at $x=0$, yields

$$y(x)\ =\ L^{-1}(4\ +\ x^5)\ -\ L^{-1}(5!\ y). \qquad (29)$$

Substituting the decomposition series $y_n(x)$ for $y(x)$into (29) gives

$$\sum_{n=0}^{\infty}y_n(x)=L^{-1}(4+x^5)-L^{-1}(5!\,y), \qquad (30)$$

$y_0=L^{-1}(4+x^5),$

$$y_{n+1}=-L^{-1}(5!\,A_n)\ ,\ \ n\geq 0. \qquad (31)$$

Using (31), the first several calculated solution components are

$$y_0=\frac{1}{120}x^5+\frac{1}{45360}x^{10},$$

$$y_1=-L^{-1}(5!\,y_0)=-\frac{1}{45360}x^{10}-\frac{1}{173365920}x^{15},$$

$$y_2=-L^{-1}(5!\,y_1)=\frac{1}{173365920}x^{15}+\frac{1}{3191839953120}x^{20},$$

$$y_3=-L^{-1}(5!\,y_2)=-\frac{1}{3191839953120}x^{20}-\frac{1}{193808521953446400}x^{25},$$

Other components can be evaluated by same the manner. We note that the components appear with opposite signs from term to another. Canceling these terms together from series solution gives the exact solution

$$y(x)=\frac{1}{5!}x^4. \qquad (32)$$

4. Adomian Decomposition strategy (The Second Adjusted)

The first differential operator L is defined by:

$$L(.)=x^{-m}\frac{d}{dx}x^{m-n}\frac{d^{n+1}}{dx^{n+1}}x^{2n+1}\frac{d^n}{dx^n}x^{-n-1}(.) \qquad (33)$$

Which gives the left side of differential equation as:

$$y^{(2(n+1))}+\frac{m}{x}y^{(2n+1)}=g(x)+f(x,y)\qquad ,n\geq 1\,,\ \ m>0 \qquad (34)$$

$y(0)=y'(0)=y''(0)=\cdots=y^{(2n+1)}(0)=0$

Where $g(x)$ and $f(x,y)$ is a real functions.

We rewrite (34) in the form

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$Ly = g(x) + f(x, y)$ (35)

The inverse operator L^{-1} is defined, as below

$$L^{-1}(.) = x^{n+1} \int_0^x \dots \int_0^x x^{-1-2n} \int_0^x \dots \int_0^x x^{n-m} \int_0^x x^m (.)dx \dots dx \quad (36)$$

Applying L^{-1} on (35) we find

$L^{-1}(Ly) = L^{-1}(g(x)) + L^{-1}(f(x, y))$

$y(x) = L^{-1}(g(x)) + L^{-1}(f(x, y))$ (37)

The Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad (38)$$

and

$$f(x, y) = \sum_{n=0}^{\infty} A_n \quad (39)$$

where the components $y_n(x)$ of the solution will be determined recurrently. Specific algorithms were seen in [10-11] formulate Adomian polynomials. The following algorithm:

$$A_0 = f\ u_0 \quad ( )$$

$$A_1 = f'(u_0)u_1$$

$$A_2 = f'(u_0)u_2 + \frac{1}{2}f''(u_0)u_1^2$$

$$A_3 = f'(u_0)u_3 + f'(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0)u_1^3, \quad (40)$$

$$\vdots$$

can be used to construct Adomian polynomials, when $f(u)$ is a nonlinear function.

By substituting (38) and (39) into (37),

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(g(x)) + L^{-1} \sum_{n=0}^{\infty} A_n \quad (41)$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$y_0 = L^{-1}(g(x))$

$y_{n+1}(x) = L^{-1}A_n \quad n \geq 0,$ (42)

Which gives

$y_0 = L^{-1}(g(x)),$

$y_1 = L^{-1}A_0\ ,$

$y_2 = L^{-1}A_1\ ,$

$\vdots \quad y_3 = L^{-1}A_2\ ,$ (43)

From (40) and (43), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (38) can be immediately obtained.

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5. Illustrative Examples

Example 1. We assume the non-linear initial value problem:

$$y^{(4)} + \frac{2}{x} y''' = 72 - x^8 + y^2, \tag{44}$$

$y(0) = 0\,,\; y'(0) = 0\,,\; y''(0) = 0.$

the exact solution is $y(x) = x^4$.

Eq.(44) can be written as

$$Ly \quad y \quad = 72 - x^8 + \quad {}^2, \tag{45}$$

Where differential operator

$$L(.) = x^{-2} \frac{d}{dx} x \frac{d^2}{dx^2} x^3 \frac{d}{dx} x^{-2} (.),$$

And inverse operator

$$L^{-1}(.) = x^2 \int_0^x x^{-3} \int_0^x \int_0^x x^{-1} \int_0^x x^{-2} (.) dxdxdxdx.$$

on both sides of (45), and using the initial conditions at x = 0, yields

$$y(x) = L^{-1}(72 - x^8) + L^{-1}(y^2), \tag{46}$$

Substituting the decomposition series $y_n(x)$ for y(x) into (46) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(72 - x^8) + L^{-1}(y^2), \tag{47}$$

$y_0 = L^{-1}(72 - x^8),$

$$y_{n+1} = -L^{-1}(A_n)\,,\, n \geq 0. \tag{48}$$

$A_0 = y_0^2\,,\, A_1 = 2y_0y_1,$

$$A_2 = y_1^2 + 2y_0y_2\,, \tag{49}$$

...

Using (49), the first several calculated solution components are

$$y_0 = x^4 - \frac{1}{14520} x^{12}\,,$$

$$y_1 = \frac{1}{14520} x^{12} - \frac{1}{943509600} x^{20} + \frac{1}{111890223244800} x^{28},$$

$$y_2 = \frac{1}{943509600} x^{20} - \frac{53}{2423542235482368} x^{28} + \cdots$$

$$y_3 = \frac{1567}{121177111774118400} x^{28} - \cdots$$

We note that:

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$$-\frac{}{14520}x^{12}+\frac{}{14520}x^{12}=0$$
$$-\frac{1}{943509600}x^{20}+\frac{1}{943509600}x^{20}=0$$
$$\frac{1}{111890223244800}x^{28}-\frac{53}{2423542235482368}x^{28}\frac{1567}{121177111774118400}x^{28}=0$$

Other components can be evaluated in a similar manner. Which gives the exact solution

$$y(x)=x^4 \tag{50}$$

Example 2. We consider the non-linear initial value problem:

$$y^{(4)}+\frac{4}{x}y'''-8(15-129x^4+49x^8+x^{12})e^{-4y}=0 \tag{51}$$

$y(0)=0,\ y'(0)=0,\ y''(0)=0,\ y'''(0)=0$ with exact solution $= ln(1+x^4)$ Eq. (51) can be written as

$$Ly=8(15-129x^4+49x^8+x^{12})e^{-4y} \tag{52}$$

Where differential operator

$$L(.)=x^{-4}\frac{d}{dx}x^3\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.)$$

And inverse operator

$$L^{-1}(.)=x^2\int_0^x x^{-3}\int_0^x\int_0^x x^{-3}\int_0^x x^4\,(.)dxdxdxdx$$

on both sides of (52), and using the initial conditions at x = 0, yields

$$y(x)=L^{-1}(8(15-129x^4+49x^8+x^{12})e^{-4y}) \tag{53}$$

Substituting the decomposition series $y_n(x)$ for y(x) into (53) gives

$$\sum_{n=0}^{\infty}y_n(x)=L^{-1}(8(15-129x^4+49x^8+x^{12})e^{-4y}), \tag{54}$$

$$y_0=0$$
$$y_{n+1}=L^{-1}(8(15-129x^4+49x^8+x^{12})A_n),\ n\geq 0 \tag{55}$$
$$A_0=e^{-4y_0},$$
$$A_1=-4y_1e^{-4y_0},$$
$$A_2=-4y_2e^{-4y_0}+8y_1^2e^{-4y_0}y_1^2,$$
$$A_3=-4y_3e^{-4y_0}+16y_1y_2e^{-4y_0}-\frac{32}{3}y_1^3e^{-4y_0}, \tag{56}$$

...

Using (56), the first several calculated solution components are

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$y_0 = 0$

$y_1 = x^4 + \frac{43}{126}x^8 + \frac{49}{2145}x^{12} + \frac{1}{7140}x^{16}$

$y_2 = -\frac{10}{63}x^8 + \frac{11266}{45045}x^{12} - \frac{560759}{10720710}x^{16} + \cdots$

$y_3 \quad = \frac{544}{9009}x^{12} - \frac{25972}{153153}x^{16} + \cdots$

$y_4 = -\frac{10096}{357357}x^{16} + \cdots$

$$y(x) = x^4 - \frac{1}{2}x^8 + \frac{1}{3}x^{12} - \frac{1}{4}x^{16} + \ \ldots \tag{57}$$

That converges to the exact solution = $ln(1 + x^4)$ by Taylor series.

Example 3. We assume the linear initial value problem:

$$y^{(6)} + \frac{1}{x}y^{(5)} = 2 - x^6 + 6!\,y\,, \tag{58}$$

$y(0) = 0 \ , y'(0) = 0 \ , \quad y''(0) = 0 \, , \quad y'''(0) = 0 \ , \ y^{(4)}(0) = 0 \, , \ y^{(5)}(0) = 0.$

With exact solution $=\frac{1}{6!}x^6$.

Eq. (58) can be written as

$$Ly = 3 - x^6 + 6!\,y\,, \tag{59}$$

Where differential operator

$$L(.) = x^{-1}\frac{d}{dx}x^{-1}\frac{d^3}{dx^3}x^5\frac{d^2}{dx^2}x^{-3}(.),$$

And inverse operator

$$L^{-1}(.) = x^3\int_0^x\int_0^x x^{-5}\int_0^x\int_0^x\int_0^x x\int_0^x x\,(.)dxdxdxdxdxdx.$$

On both sides of (59), and using the initial conditions at $x = 0$, yields

$$y(x) = L^{-1}(2 - x^6) + L^{-1}(6!\,y). \tag{60}$$

Substituting the decomposition series $y_n(x)$ for $y(x)$into (60) gives

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(2 - x^6) + L^{-1}(6!\,y), \tag{61}$$

$y_0 = L^{-1}(2 - x^6),$

$$y_{n+1} = L^{-1}(6!\,A_n)\,, n \geq 0. \tag{62}$$

Using (62), the first several calculated solution components are

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$$y_0 = \frac{}{720}x^6 - \frac{}{760320}x^{12},$$
$$y_1 = L^{-1}(6!\,y_0) = \frac{1}{760320}x^{12} - \frac{1}{15200317440}x^{18},$$
$$y_2 = L^{-1}(6!\,y_1) = \frac{1}{15200317440}x^{18} - \frac{1}{2153580974899200}x^{24},$$
$$y_3 = L^{-1}(6!\,y_2) = \frac{1}{2153580974899200}x^{24} - \frac{1}{1329892245105603379200}x^{30},$$
...

Other components can be evaluated by same the manner. We note that the components appear with opposite signs from term to another. Canceling these terms together from series solution gives the exact solution

$$y(x) = \frac{1}{6!}x^5. \qquad (63)$$

6. Conclusion

In this work, we have used the adjusted Adomian decomposition strategy for solving singular initial value problems of higher odd-order. We have presented a two new differential operator for solving this problems. We have demonstrated that the strategy is quick convergent for solving IVPs. The given examples illustrate the advantages of using the proposed method in this work for these kinds of equations. Finally the adjusted Adomian decomposition strategy is effective and efficient in finding the analytical solutions for a wide class of initial value problems.

References

Adomian, G. (1988). A review of the decomposition in applied mathematics, Mathematical analysis and applications., 135,501-544.

Adomian, G. (1991). A review of the decomposition method and some recent results for nonlinear equations, Computers Math. Applic. 21,101-127.

Adomian, G. (1994). Solving FrontierProblems of Physics:The Decomposition Method, Kluwer academic publishers, London.

Duan , J. (2015). The Adomian Polynomials and the New Modified Decomposition Method for BVPs of nonlinear ODEs , Mathematical Computation March 4, 1 6.

Fowler, R. H. (1931). Further studies of Emdens and similar differential equations, Quarterly Journal of Mathematics, 2(1), 259-288.

Hasan, Y. Q. & Zhu, L. M . (2008). Modified Adomian decomposition method for singular initial value problems in the second-order ordinary differential equations, Surveys in mathematics and its applications, 3, 183-193.

Hasan, Y. Q. (2012). Modified Adomian decomposition method for second order singular initial value problems. Advances in Computational Mathematics and its Applications, 1, 94-99.

Hasan, Y. Q. (2014). A new development to the Adomian decomposition for solving singular IVPs of Lane-Emden Type. United States of America Research Journal (USARJ) 2, 2332-2160.

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Hasan, Y. Q. & Mutaish, A.M.S. (2017).Modified Adomian decomposition method for solving higher odd order boundary value problems. MAYFEB Journal of Mathematics, 2,74-79.

Wazwaz, A. M. (1997). A First Course in Integral Equations, World Scientific, Singapore.

Wazwaz, A. M. (2000). A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math.Comput. 111 (1), 33-51.

Wazwaz, A. M. (2002). A new method for solving singular initial value problems in the second-order ordinary differential equations. Applied Mathematics and Computation. 128, 45-57.

Wazwaz, A. M., Rach,R., Bougoffa L. & Duan, J.S. (2014). Solving the Lane–Emden Fowler Type Equations of Higher Orders by the Adomian Decomposition Method, CMES, 100,507-529.