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ASSESSING EFFECTIVENESS: CONVERGENCE ORDER, CONSISTENCY, AND STABILITY IN NEWLY SUGGESTED APPROACHES

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Abstract: Mathematical modeling serves as a vital tool for interpreting real-world scenarios through the lens of mathematical symbols and relationships, commonly applied across various disciplines including Sciences and Engineering. Through the development of models, researchers aim to gain insights into complex physical phenomena, often resulting in the formulation of differential equations containing derivatives of unknown functions. These equations, termed as Differential Equations, serve as foundational components in understanding phenomena spanning from physical sciences to fields as diverse as Economics, Medicine, Psychology, and Operation Research, extending even into domains like Biology and Anthropology. However, the quest for analytical solutions to these differential equations, stemming from real-life modeling endeavors, often presents formidable challenges. Many equations arising from such modeling efforts defy straightforward analytical solutions, necessitating the exploration of alternative methods for their resolution.

This abstract underscores the ubiquity of differential equations across numerous disciplines and emphasizes the significance of mathematical modeling in advancing understanding and problem-solving capabilities. It highlights the interdisciplinary nature of differential equations, transcending traditional boundaries and finding applications in diverse fields. Moreover, it acknowledges the inherent complexity associated with obtaining analytical solutions to these equations, paving the way for the exploration of alternative solution strategies.

Keywords: Mathematical modeling, Differential equations, Interdisciplinary applications, Analytical solutions, Problem-solving.

INTRODUCTION

Mathematical model is a means of translating real life situations into mathematical symbols and relations. This concept is commonly used in Sciences and Engineering. Models are developed to help in understanding of physical phenomena. These models frequently resulted in equations that contain derivatives of unknown function of one or several variables. These types of equations are referred to as Differential Equations. Differential equations are not only encountered in physical sciences, but also in diverse fields like Economics, Medicine,

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Psychology, and Operation Research and even in areas such as Biology and Anthropology. In reality, the analytical solutions of some of the equations arising from modelling of real life situations might not be easily obtained. This necessitated the need for approximate solution by the application of numerical methods. To that extent, several algorithms have been proposed in literature based on the nature and the type of the differential equations to be solved such as Wambecq (1976), Jain (2003), Davis (2013), Qureshi et al. (2013), Fadugba and Falodun (2017), Fadugba and Okunlola (2017), just to mention a few. This paper is motivated by the work of Qureshi and Fadugba (2018). In this paper, the order of convergence, consistency and stability properties of the FadugbaFalodun and Fadugba-Okunlola schemes were investigated. The TOOSS and SOOSS have been applied on IVPs of first and second order ODEs (Fadugba and Ajayi, 2017; Qureshi and Fadugba, 2018; Fadugba, 2019; Fadugba, 2020).

Therefore, Equation 4 becomes:

$LTE(TOOSS) \square y(x_n) \square hf(x_n, y(x_n)) \square \frac{1}{} h^1 f^{(1)}(x_n, y(x_n))$

ORDER OF CONVERGENCE OF THE SCHEMES

$\frac{1}{3!} h^3 f^{(2)}(x_n, y(x_n)) \square \frac{1}{4!} h^4 f_2^{(3)}(x_n, y(x_n)) \square O(h^5)$

$\square h$

In order to check the order of convergence of the schemes, the formula of the schemes is subtracted from the Taylor's series expansion for $y(x)$ in powers of h

$\square y_n \square 1^8 \square\square\square\square 1 \square 2h \square \frac{(2h)^2}{2!} \square \frac{(2h)^3}{3!} \square \frac{(2h)^4}{4!} \square \ldots \square 1 \square\square\square\square f^{(2)}(x_n, y_n)$

under the localizing assumptions. The convergence of $h \square f(x, y) 1 f^{(2)}(x, y) \square h^2 \square f^{(1)}(x, y) 1 \; f^{(2)}(x, y) \square$ Thus,

Order of convergence of TOOSS

$LTE(TOOSS) \square y(x_n) \square hf(x_n, y(x_n)) \square \frac{1}{2} h^2 f^{(1)}(x_n, y(x_n))$

From the Taylor's series, one obtains:

$\square \frac{1}{3!} h^3 f^{(2)}(x_n, y(x_n)) \square \frac{1}{4!} h^4 f^{(3)}(x_n, y(x_n)) \square O(h^5)$

$\frac{1}{2} h^2 y''(x_n) \square \frac{1}{3!} h^3 y'''(x_n) \square \frac{1}{4!} h^3 y^{iv}(x_n) \square O(h^5) \quad \square \square\square y_n \square hf(x_n, y_n) \square \frac{h^2}{2} f^{(1)}(x_n, y_n) \square \frac{h^3}{3!} f^{(2)}(x_n, y_n) \square \frac{1}{2} h^4 f^{(2)}(x_n, y_n) \square\square \square y(x_n \square h) \square y(x_n) \square hy'(x_n) \square h$

$\square y(x_n) \square hf(x_n, y(x_n)) \square \frac{1}{2} h^2 f^{(1)}(x_n, y(x_n))$

$\frac{1}{3!} \square h^3 f^{(2)}(x_n, y(x_n)) \square \square \frac{1}{4!} h^4 f^{(3)}(x_n, y(x_n)) \square O(h^5)$

^{1} h $1 \square 2h \square \frac{(2h)^2}{2!} \square \frac{(2h)^3}{3!} \square \frac{(2h)^4}{4!} \square \ldots$ (5) $e \square$

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$$(1)$$

Fadugba and Okunlola (2017) derived a scheme of the form:

$$y_{n+1} = y_n + \frac{1}{8}(e^{2h} - 1) f^{(2)}(x_n, y_n) + h\left[f(x_n, y_n) - \frac{1}{4} f^{(2)}(x_n, y_n)\right] + \frac{h^2}{2}\left[f^{(1)}(x_n, y_n) - \frac{1}{2} f^{(2)}(x_n, y_n)\right] \quad (2)$$

Using Equations 1 and 2, the local truncation error becomes:

$$LTE(TOOSS) = y(x_n + h) - y_{n+1} \quad (3)$$

$$LTE(TOOSS) = y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2}h^2 f^{(1)}(x_n, y(x_n)) + \frac{1}{3!}h^3 f^{(2)}(x_n, y(x_n)) + \frac{1}{4!}h^4 f^{(3)}(x_n, y(x_n)) + O(h^5)$$
$$- \left[y_n + \frac{1}{8}(e^{2h} - 1) f^{(2)}(x_n, y_n) + h\left[f(x_n, y_n) - \frac{1}{4} f^{(2)}(x_n, y_n)\right] + \frac{h^2}{2}\left[f^{(1)}(x_n, y_n) - \frac{1}{2} f^{(2)}(x_n, y_n)\right] \right] \quad (4)$$

Replacing the term e^{2h} in Equation 4 by its Maclaurin's series given by

$$(7)$$

Under the local assumptions, the terms up to h^3 have been cancelled, then Equation 7 becomes:

$$LTE(TOOSS) = \frac{1}{12} h^4 \left[\frac{1}{2} f^{(3)}(x_n, y(x_n)) - f^{(2)}(x_n, y(x_n))\right] + O(h^5) \quad (8)$$

Hence, the TOOSS has the convergence of third order.

Order of convergence of SOOSS

The Taylor's series expression for $y(x)$ in powers of h is given by:

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(x_n) + \frac{1}{3!}h^3 y'''(x_n) + O(h^4)$$
$$= y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2}h^2 f^{(1)}(x_n, y(x_n)) + \frac{1}{3!}h^3 f^{(2)}(x_n, y(x_n)) + O(h^4)$$

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(9)

Fadugba and Falodun (2017) derived a scheme of the form:

$$y_{n+1} = y_n + h(f(x_n, y_n) - f^{(1)}(x_n, y_n)) + (e^{h} - 1) f^{(1)}(x_n, y_n) \quad (10)$$

The local truncation error is given by

$$LTE(SOOSS) = y(x_n + h) - y_{n+1} \quad (11)$$

Using Equations 9 and 10, Equation 11 becomes:

$$LTE(SOOSS) = y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2} h^2 f^{(1)}(x_n, y(x_n)) + \frac{1}{3!} h^3 f^{(2)}(x_n, y(x_n)) + O(h^4) - \left[y_n + h(f(x_n, y_n) - f^{(1)}(x_n, y_n)) + (e^{h} - 1) f^{(1)}(x_n, y_n)\right] \quad (12)$$

Using the Maclaurin's series expansion of e^{h} and simplifying the Equation 12 under the localizing assumption, one gets:

$$LTE(SOOSS) = \frac{h^3}{3!}\left[f^{(1)}(x_n, y_n) - f^{(2)}(x_n, y_n)\right] + O(h^4) \quad (13)$$

Hence, SOOSS has the convergence of second order.

Remark

The analysis of local truncation error indeed determines the order of convergence for any numerical technique designed to solve IVPs in ODEs.

CONSISTENCY PROPERTIES OF THE SCHEMES

It is a known fact that any numerical method having an order of accuracy greater than or equal to 1 is considered to be consistent. In other words, a numerical integration method is said to be consistent if it has at least order p =1.

Consistency analysis of TOOSS

Among many, one of the ways to analyze the consistency of a numerical technique is to check that whether (Qureshi and Fadugba, 2018):

$$\lim_{h \to 0} \left(\frac{LTE(TOOSS)}{h} \right) = \lim_{h \to 0} \left(\frac{\frac{1}{12} h^4 \left[12 f^{(3)}(x_n, y(x_n)) - f^{(2)}(x_n, y(x_n))\right] + O(h^5)}{h} \right) = 0 \quad (14)$$

From Equation 14, it is observed that TOOSS has consistency characteristics.

Consistency analysis of SOOSS

Following the same procedures as that of the Qureshi and Fadugba (2018), one obtains that:

$$\lim_{h \to 0} \left(\frac{LTE(SOOSS)}{h} \right) = \lim_{h \to 0} \left(\frac{\frac{1}{3!} h^3 \left[f^{(2)}(x_n, y(x_n)) - f^{(1)}(x_n, y(x_n))\right] + O(h^4)}{h} \right) = 0 \quad (15)$$

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From Equation 15, it is observed that SOOSS has consistency characteristics.

Remark

For a numerical technique to be consistent, it is important for the truncation errors to be zero when the step size h gets smaller and ultimately reaches to zero.

STABILITY ANALYSES OF THE SCHEMES

A numerical integration method is said to be stable if it is capable of damping out the small fluctuations carried out in the input data. A one step explicit numerical integration method is reserved to be stable if a small perturbation in the initial conditions of the IVP leads to a small perturbation in the following numerical approximation.

Stability analysis of TOOSS

For the stability analysis of TOOSS, one of the popular ways is to apply the scheme to the Dahlquist's test equation:

$$y'(x) = \lambda y(x), \; y(0) = 1, \; \lambda < 0 \qquad (16)$$

whose exact solution is given by

$$y(x) = \exp(\lambda x) \qquad (17)$$

where $\lambda < 0$ is, in general, a constant. For an integration interval $[x_n, x_{n+1}]$, where

$$h = x_{n+1} - x_n \qquad (18)$$

The exact solution at the point

$$x = x_{n+1} \qquad (19)$$

is obtained as

$$y(x_{n+1}) = \exp(\lambda x_{n+1}) = y(x_n)\exp(\lambda h) \qquad (20)$$

The numerical approximation obtained using TOOSS gives

$$y_{n+1} = \left[1 + \lambda h + \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!}\right] y_n \qquad (21)$$

Setting

$$A = 1 + \lambda h + \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!} \qquad (22)$$

Equation 21 becomes

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$y_{n□1}$ □Ay_n (23) 4 Afr. J. Math. Comput. Sci. Res.

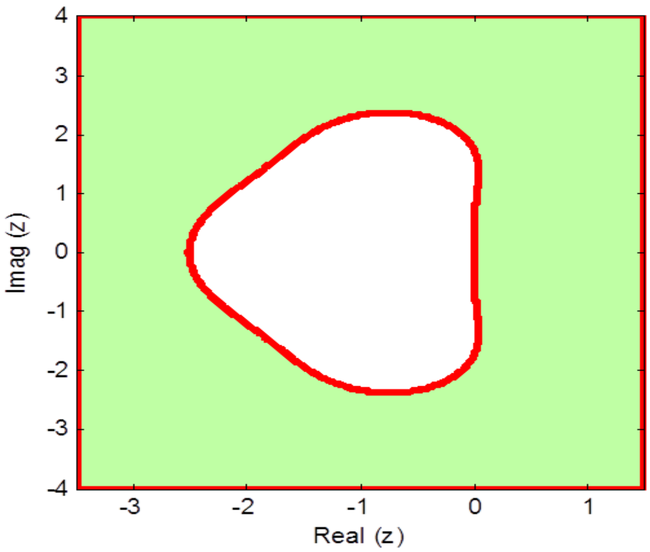

Figure 1. The stability region (unshaded) of TOOSS.

Comparison of Equations 20 and 23 shows that the factor A is merely an approximation for the factor exp(□h)in the exact solution. Truly, the factor A is the four-term approximation for the Maclaurin's series for exp(□h) for small□h . The error growth factor A can be controlled by A □1so that the errors may not magnify. Thus, the stability of TOOSS requires that

z2 ‖ z3 A □ 1□ z□‖□ □1, z□□h (24) 2! 3!

Using Equation 24, the stability region is plotted in Figure 1. Hence, TOOSS is found to be stable in Figure 1.

Stability analysis of SOOSS

To discuss the stability analysis of SOOSS, consider the following Dahlquist's test equation of the form:

y□(x) □□y(x), y(0) □1,□□ 0 (25)

The exact solution of Equation 25 is given by

y(x)□exp(□x) (26)

For an integration interval[x_n,$x_{n□1}$], where h □ $x_{n□1}$ □ x_n and following the procedures of Qureshi and Fadugba (2018); the exact solution at the point x□$x_{n□1}$ is obtained as:

y($x_{n□1}$) □ y(x_n)exp(□h) (27)

When applied SOOSS on this test problem, one gets:

$y_{n□1}$ □By_n (28)

where

B□1□□h□ _____(□h)2 (29)

2!

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Comparing Equations 27 and 28, it is clearly seen that Equation 28 is a three-term approximation for the function $e^{\lambda h}$ in the exact solution. The error growth factor given by Equation 28 can be controlled by $B < 1$ so that the errors may not amplify. Thus, the stability function of SOOSS requires that:

$$\left| 1 + \lambda h + \frac{(\lambda h)^2}{2!} \right| < 1 \qquad (30)$$

Setting

$$z = \lambda h \qquad (31)$$

Therefore, Equation 30 becomes:

$$\left| 1 + z + \frac{z^2}{2!} \right| < 1 \qquad (32)$$

The region of absolute stability for SOOSS is defined by the region in the complex plane such that $\left| 1 + z + \frac{z^2}{2!} \right| < 1$. The stability region is plotted in Figure 2.

Remark

The notion of stability may be taken in different contexts: it may be associated with the specific numerical technique used, or the step size h used in numerical computations or with the particular problem being solved.

NUMERICAL EXAMPLES AND DISCUSSION

Here presents the implementation of the two schemes on IVPs of stiff differential equations. The discussion of results is also presented. All the calculations were carried out via MATLAB R2014a, Version: 8.3.0.552, 32 bit (win 32) in double precision.

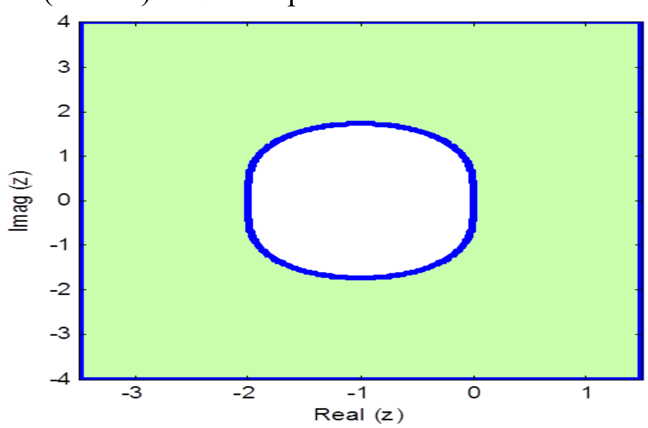

Figure 2. The stability region (unshaded) of SOOSS.

Example 1

$y' = -100(y - x) + 1,\ y(0) = 1,\ x \in [0, 0.1],\ y(x) = \exp(-100x) + x$

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Example 2

$$y' = -20y + 24,\ y(0) = 0,\ x \in [0,0.1],\ y(x) = \frac{6}{5}(1 - \exp(-20x))$$

Example 3

$$y' = -10(y - x)^2,\ y(0) = 2,\ x \in [0,0.1],\ y(x) = 1 + \frac{1}{10x+1}$$

The final absolute relative errors at $x = b$ defined by

FABRE $= \|y(b) - y_N\|$ generated via SOOSS and TOOSS for Examples 1 to 3 are shown in Tables 1 to 3, respectively. The plots of the Tables 1 to 3 were displayed in Figures 3 to 5, respectively.

It is observed from Tables 1 to 3 that both SOOSS and TOOSS perform excellently and yield smaller error for every decreasing step length, h. It is also observed from Tables 1 to 3 that the order of accuracy of SOOSS and TOOSS have been confirmed when applied to stiff differential equations taking the step length h having a first order decrease in its magnitude, that is $h = 10^{-2}, 10^{-3}$, Fadugba 5

Table 1. The FABRE via SOOSS and TOOSS for Example 1. 10-4, 10-5, 10-6. It is clearly seen in Tables 1 to 3 for every decrease in h, there are second order and third-order decrease in the magnitude of the computed FABRE via SOOSS and TOOSS, respectively. It is observed from the Figures 3 to 5 that the FABREs generated via SOOSS are greater than that of the TOOSS.

h	**SOOSS**	**TOOSS**
10-2	0.00089917	0.00002888
10-3	0.00000081	0.00000002
10-4	0.00000001	0.00000000
10-5	0.00000000	0.00000000
10-6	0.00000000	0.00000000

Table 2. The FABRE via SOOSS and TOOSS for Example 2.

h	**SOOSS**	**TOOSS**
10-2	0.00240159	0.00014029
10-3	0.00002088	0.00000012
10-4	0.00000021	0.00000000
10-5	0.00000000	0.00000000
10-6	0.00000000	0.00000000

Table 3. The FABRE via SOOSS and TOOSS for Example 3.

h	**SOOSS**	**TOOSS**
10-2	0.00139160	0.00012157
10-3	0.00001210	0.00000010
10-4	0.00000012	0.00000000

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10-5	0.00000000	0.00000000
10-6	0.00000000	0.00000000

Conclusion

In this paper, notes on the order of convergence, consistency and stability properties of TOOSS and SOOSS have been successfully presented. It is observed that TOOSS and SOOSS have the convergence of third order and second order, respectively. From the analysis, it is observed that the two methods are convergent, consistent since they have order of accuracy greater than 1. It is also observed that TOOSS and SOOSS are stable as shown in Figures 1 and 2, respectively. Hence, it can be concluded from the numerical results that TOOSS performs better than SOOSS since it has a higher order of accuracy.

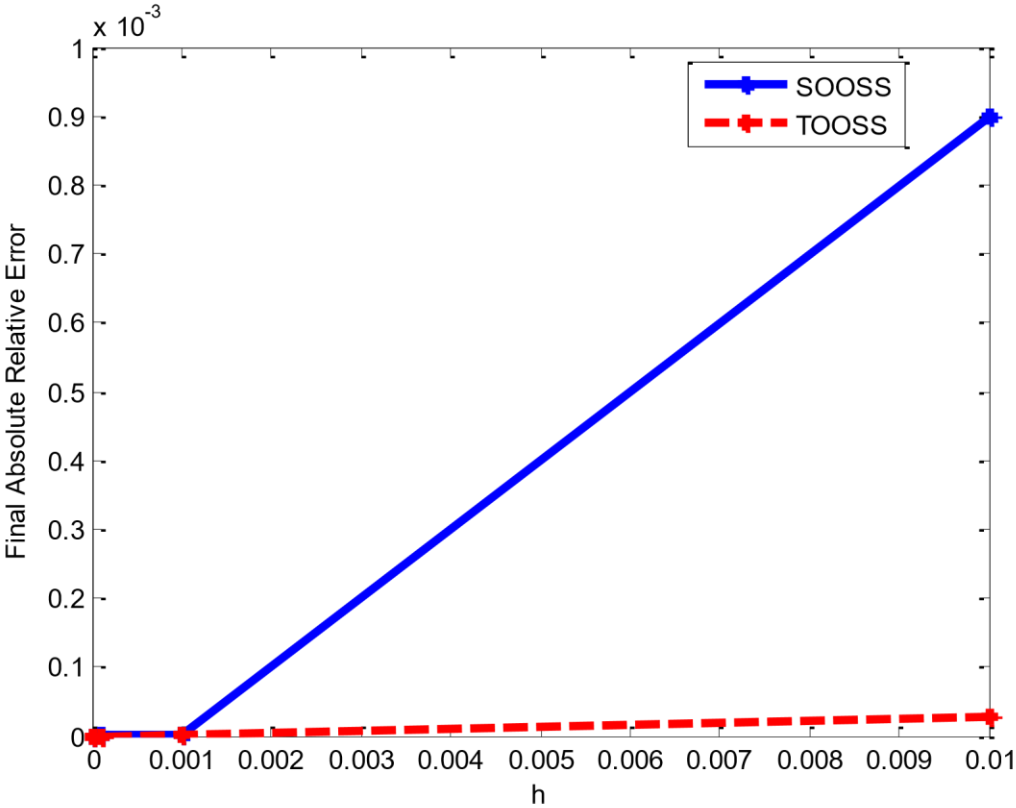

Figure 3. The plot of the FABRE generated via SOOSS and TOOSS using Table 1.

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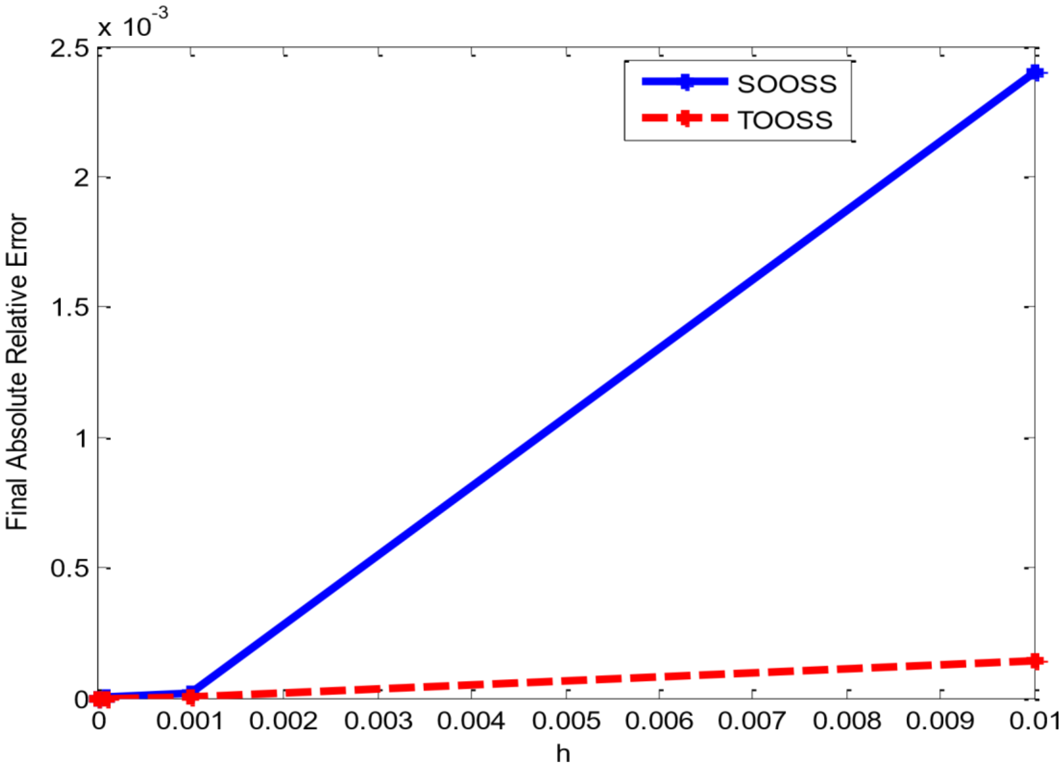

Figure 4. The plot of the FABRE generated via SOOSS and TOOSS using Table 2.

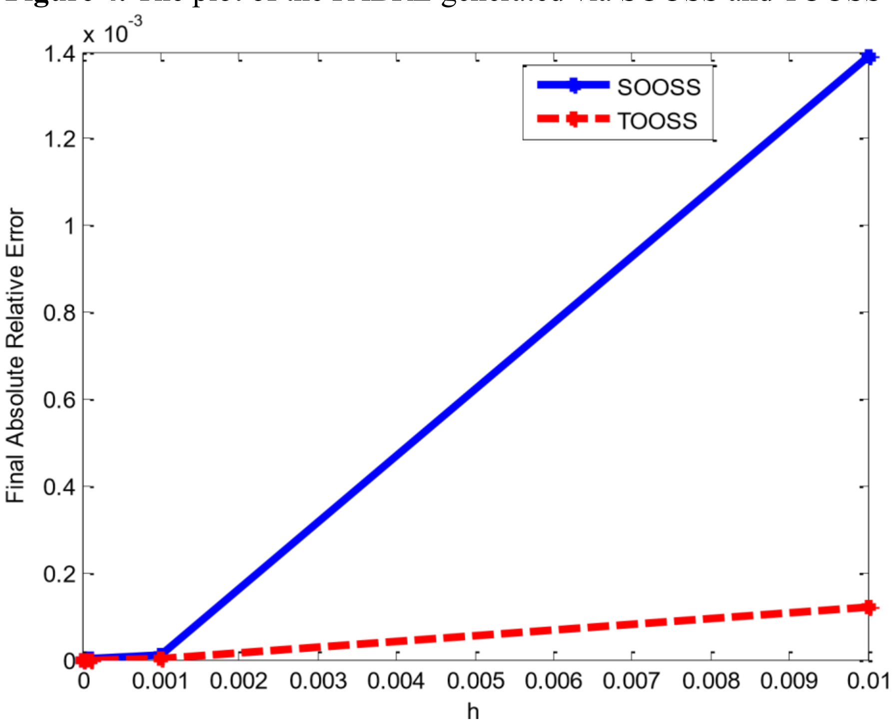

Figure 5. The plot of the FABRE generated via SOOSS and TOOSS using Table 3.

CONFLICT OF INTERESTS

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The author has not declared any conflict of interests.

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