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EXPLORING THE CORDIALITY OF DUPLICATED GRAPHS

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Abstract: Cordial graph theory has provided valuable insights into graph labeling, particularly through the concept of cordiality. Initially introduced as a weaker alternative to graceful and harmonious graphs, cordial graphs are characterized by {0, 1} binary vertex labeling. This abstract explores various properties of cordial graphs, including the relationship between cordiality and graph structures, such as trees and wheels. Notably, the cordiality of Eulerian graphs is also addressed in connection to its size congruence. While cordial graphs have been a topic of interest, this abstract serves as an introduction to the field and its fundamental results.

Keywords: Cordial graphs, binary labeling, graph structures, Eulerian graphs, graph theory.

1. Introduction

By a graph, we mean a finite undirected graph without loops and multiple edges. For terms not defined here, we refer to Harary [7]. Cordial graph was first introduced by I. Cahit [1] in 1987 as a weaker version of graceful and harmonious graphs and was based on {0, 1} binary labeling of vertices. He showed that (i) every tree is cordial (ii) is cordial if and only if $n \leq 3$ (iii) , is cordial for all r and s (iv) the wheel is cordial if and only if $n \equiv 3(mod4)$ (v) is cordial if and only if $n \not\equiv 2(mod4)$ (vi) an Eulerian graph is not cordial if its size is congruent to 2 modulo 4. Other types of cordial graphs are considered in [3, 7, 10, 11]. For more related results on cordial graphs, one can refer to Gallian [6].

Definition 1.1 [1]

A binary vertex labeling of graph G(V, E), where each edge uv is labeled with |f(u) – f(v)| (mod 2), is called a ***cordial labeling*** if (0)– (1) ≤ 1 and | (0) – (1)| ≤ 1, where ( )denote the number of vertices labeled with under f and ( ) denote the number of edges labeled with i, where i = 0, 1. A graph G is called cordial if it admits a cordial labeling.

Definition 1.2 [15]

Let be a graph with vertex set V(G) and edge set E(G) and let : ( ) → {0, 1}. Define on V(G) by *( ) = ∑ { ( ), ( )}(2). The function f is called an ***E-cordial labeling*** of G if | (0) − (1)| ≤ 1 and | (0) − (1)| ≤ 1. A graph is called ***E-cordial*** if it admits E-cordial labeling.

In 1997 Yilmaz and Cahit [15] have introduced E-cordial labeling as a weaker version of edge-graceful labeling. More results are seen in [15,4]

Definition: 1.3 [4]

A ***prime cordial labeling*** of a graph G with vertex set V is a bijection f from V to {1, 2, 3, . . . , |V|} such that if each edge uv is assigned the label 1 if gcd (f(u), f(v)) = 1 and 0 if gcd (f(u), f(v)) > 1, then the number of edges having label 0, and the number of edges having label 1,differ by at most 1. Sundaram et. al [9] has introduced the notion of prime cordial labeling and proved some graphs are prime cordial

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Definition: 1.4
The ***fan F_n*** is the graph obtained by taking (n - 2) concurrent chords in cycle . The vertex at which all the chords are concurrent is called the ***apex vertex***. More precisely, + K is called the ***fan F_n.***
Definition: 1.5
A ***helm*** , ≥ 3 is the graph obtained from the wheel by adding a pendant edge at each rim vertex.
Definition: 1.6
The ***flower*** is the graph obtained from a helm by joining each pendant vertex to apex vertex of the helm.

Definition: 1.7
The ***closed helm*** is the graph obtained from a ***helm*** by joining each pendant vertex to form a cycle.
Definition 1.8[11]
Let G = (V, E) be a simple graph and f: V→ {1, 2, 3, . . . |V|} be a bijection. For each edge uv, assign the label 1 if either f(u) / f(v) or f(v) / f(u) and the label 0 otherwise. Then f is called a ***divisor cordial labeling*** . A graph with a divisor cordial labeling is called ***divisor cordial graph*.**
R.Varatharajan, S.Navaneethakrishnan and K.Nagarajan [13], introduced the concept of divisor cordial and proved the graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial graphs and in [14], they proved some special classes of graphs such as full binary tree, dragon, corona, ∗ , and ∗ , are divisor cordial. We proved in [8] that some special graphs such as switching of a vertex of cycle, wheel, helm, duplication of arbitrary vertex of cycle, duplication of arbitrary edge of cycle, split graph of , , , , , are divisor cordial graphs.
Labeled graph have variety of application in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal auto correlation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication networks and to determine optimal circuit layouts.
In this paper, we prove that fan graph , switching of a pendant vertex of a helm graph , switching of a vertex of flower graph , switching of closed helm graph and also duplication of a arbitrary vertex by an edge of a fan are divisor cordial.

2. Main results

Theorem: 2.1
The fan is a divisor cordial graph.

Proof:
Let v be the apex vertex and , , , … , be the other vertices of the path in fan . Thenp = n + 1 and q = 2n –1. We define vertex labeling f: V→ {1,2, . . ., p} as follows.

( ) = 1
( ) = + 1 ; 1 ≤ ≤

Since 1 divides any integers, the edges receive label 1 and the consecutive numbers does not divide each other, so the (n –1) edges in path receive label 0.
Now we observe that (0) = − 1; (1) = .
Hence (0) − (1) = 1.
Thus is a divisor cordial graph. ■

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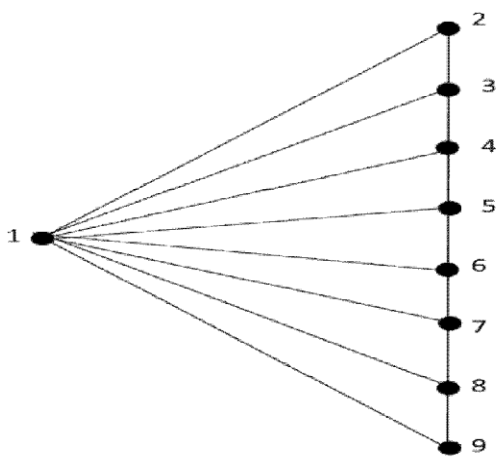

Figure 1: is a divisor cordial

3. Switching of a vertex

Definition: 3.1[13]

*A **vertex switching G_v*** of a graph G is the graph obtained by taking a vertex v of G, removing the entire edges incident to v and adding edges joining v to every other vertex which is not adjacent to v in G.

Theorem: 3.2

Switching of a pendant vertex in helm graph admits divisor cordial labeling. **Proof:**

Let be a helm graph with w as the apex vertex and , , , . . . be the pendant vertices and . . .Be the vertices of cycle. Let denote the graph obtained by switching of a pendant vertex of = . Here p = 2n + 1 and q = 5n – 2. Without loss of generality, let us assume that the vertex is switched. We consider two cases.

( ) = 1

( ) = 2

( ) = 2 + 1 ; 1 ≤ ≤ −
2(+ 1); − + 1 ≤ ≤ − 1

( ) = 2(+ 1); 1 ≤ ≤ −
2 + 1; − + 1 ≤ ≤ − 1

Since the pendant vertex is given label 1, the 2(n - 1) + 1 edges incident to it receives label 1. Also, since the apex vertex is given label 2 and cyclic vertex ; 1 ≤ ≤ is labeled with even integers, they divide each other and receive label 1. All other edges receive label 0. That is, 2(− 1) + 1 + edges receive label 1.

Hence (1) = (0) = ____

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Therefore, | (1) − (0)| = 0

Case (ii) n is odd

( ) = 1
( ) = 2
( ) = ; –2(+ 1) 1 ≤ ≤ ⌊ ⌋
2 + 1 ; ⌈−⌉ ≤ ≤

( ) = 2 + 1 ; 1 ≤ ≤ ⌊−⌋
2(+ 1); ⌈−⌉ ≤ ≤

As the above case, we observe that 2 − 1 + ⌈−⌉ edges receive label1.

That is, (1) = ___ edges receive label 1and (0) = ___ edges receive label 0

Hence, (0) − (1) = 1

From both the cases | (0) − (1)| ≤ 1

Hence switching of a pendant vertex of is a divisor cordial graph. ■

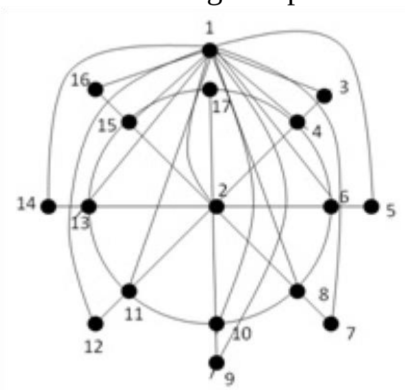

Figure 2: Switching of a pendant vertex in

Note: 3.3

Switching of a vertex of inner cycle of a helm , the graph becomes disconnected.

Theorem: 3.4

The graph obtained by switching of a vertex of flower graph is divisor cordial.

Proof:

Let w be the apex vertex; , , ,… , be the vertices of cycle and , , , . . . , be the pendant vertices. We define f: V(G) → {1, 2, . . . ,p}

Case (i) Switching of a vertex ,≤ ≤ .

Without loss of generality let us assume that the rim vertex is switched. Then p = 2n + 1, q = 3(2n – 1)

( ) = 1
( ) = 2
( ) = 2 (+ 1); 1 ≤ ≤ − 1
( ) = 2 + 1 ; 1 ≤≤ – 1

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( ) = ( ) + 2

Since is given label 1 the edges adjacent to receives label 1 the apex vertex is labeled with 2, and ; 1 ≤ ≤ is labeled with even integers, since they divide each other, that edges receive label 1. Hence (2n – 1 + n) edges label 1 and other (3n – 1) edges receive label 0.

That is, (1) = (0) = (3 − 1)

Therefore, (0) − (1) = 0

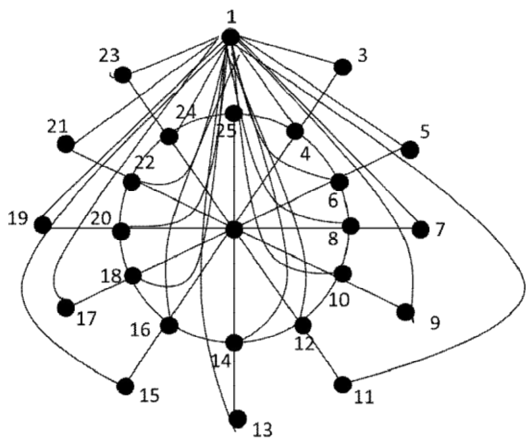

Figure 3: Switching of a pendant vertex of *Fl*12

Case (ii) Switching of a vertex ,≤ ≤

Let us assume that the vertex is switched. Here p = 2n + 1and q = 6n - 8.

( ) = 1

( ) = 2 ; 1 ≤ ≤

( ) = 2 + 1 ;1 ≤ ≤

Since the apex vertex is given label 1, the (2n - 1) edges incident to it receives label 1. The switched vertex is given label2, and so the (n - 3) vertices incident to v also receives label 1. Hence (n - 3) + (2n - 1) = 3n - 4 edges receives label 1 and other (3n - 4) edges receives label 0

That is, (1) = (0) = (3 − 4)

Therefore, | (0) − (1)| = 0

From both the cases | (0) − (1)| ≤ 1. Hence graph obtained by switching of a vertex of flower graph is divisor cordial.

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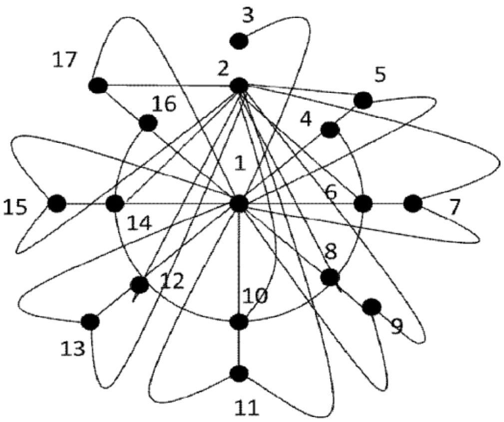

Figure: 4: switching of a vertex in

Case (iii) Switching of the apex vertex w, the graph becomes disconnected.■

Theorem: 3.5

The graph obtained by switching of a vertex of a closed helm graph is integer cordial.

Proof:

Let w be the apex vertex; be the vertices of inner cycle and , , . . . be the vertices of outer cycle.

Define f : V→ {1,2, . . .,p}

Case (i) Switching of a vertex ; ≤ ≤ .

Here p = 2n + 1 and q = 6(n - 1).Without loss of generality let us assume the vertex is switched

The labeling is as follows:

( ) = 2 ; ( ) = 2 + 1

( ) = 2(+ 1); 1 ≤≤ − 1

( ) = 2 − 1; 1 ≤ ≤

Since the vertex is given label 1, the (2n - 3) edges incident to it receive label 1. Similarly the apex vertex w is given label 2 and the vertices of inner cycle are given even integers and so (n - 2) edges and the edge ( ) receives label 1. Other edges receive label 0.

That is, (1) = (0) = 3(− 1)

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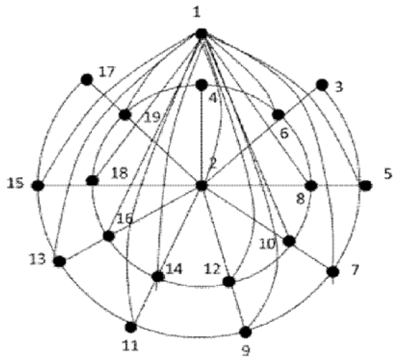

Figure: 5 Switching of in

Case (ii) Switching of a vertex ; ≤ ≤

Without loss of generality let us assume the vertex is switched. Here p = 2n + 1 and q = 6n – 8

The labeling is as follows:

( ) = 2; ( ) = 1; ( ) = 2 + 1

( ) = 2 ; 2 ≤ ≤

( ) = 2 − 1 ;2 ≤ ≤ . Now interchange and .

From the above labeling, we observe that (3n - 4) edges receive label 1 and (3n - 4) edges receive label 0.

That is, (1) = (0) = 3 − 4.

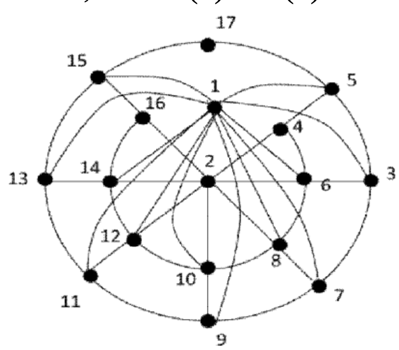

Figure: 6 Switching of vertex .

Case (iii) Switching of apex vertex w

Let the apex vertex w be switched. Then p = 2n + 1 and q = 4n.

Subcase (i) n≢ 2 (mod 4)

The labeling is as follows:

( ) = 1

The vertices of inner cycle , , . . . and rim vertices , , . . . be labeled in the following order

2, 2 x 2, 2 x2 , … 2 x2

3, 3 x 2, 3 x 2 , . . . 3 x 2 ….. ….. ….. (2)

5, 5 x 2, 5 x2 , . . . 5 x2

. . . .

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Where (2 − 1) 2 ≤ and ≥ 1, > 0. We observe that (2 − 1)2 divide (2− 1) 2 ; (<) and (2 − 1) 2 does not divide (2m+1). Now interchange the labels of and .
The remaining pendant vertices are labeled continuously other than the above labels.
From the labeling, we observe that 2n edges receives label 1 and 2n edges receive label 0.
That is, (1) = (0) = 2.

Subcase (ii) n ≡ 2 (mod 4)

The labeling is as follows:

f(w)=1

The vertices of inner cycle , , . . . and pendant vertices , , . . . be labeled in the following order

2, 2 x 2, 2 x 2 , … 2 x2
3, 3 x 2, 3 x 2 , . . . 3 x 2 ….. ….. ….. (2)
5, 5 x 2, 5 x2 , .. . 5 x2 . . . .

Where (2 − 1) 2 ≤ and ≥ 1, > 0. We observe that (2 − 1)2 divide (2 − 1) 2 ; (<) and (2 − 1) 2 does not divide (2m+1).
The remaining pendant vertices are labeled continuously other than the above labels.
From the labeling we observe that 2n edges receives label 1 and 2n edges receive label 0.

That is, (1) = (0) = 2 .

From all the cases | (0) − (1) | = 0

Hence the graph obtained by switching a vertex is divisor cordial. ■

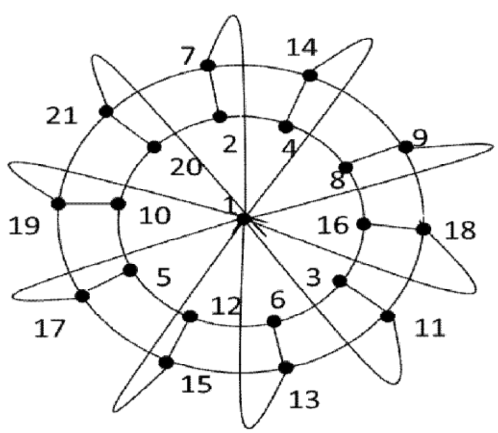

Figure: 6 switching of apex vertex in

4. Duplication of a vertex and duplication of an edge.

Definition: 4.1 [**12**]

Duplication of a vertex by a new edge = in a graph G produces a new graph such that (’) = { , ”} and (”) = { , ’}

Definition: 4.2**[12]**

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Duplication of an edge = by a vertex in a graph G produces a new graph such that (’) = { , }.

Theorem: 4.3

Duplication of a vertex by an edge in a fan graph is divisor cordial graph.

Proof:

Let v be the apex vertex and … be the other vertices of fan and let and be the newly added vertex. Then p = n + 3 and q = 2(n + 1). We define vertex labeling as f: V →{1,2,…, p}as follows.

Case (i)

Let the apex vertex v be duplicated.

Subcase (i) n is even

f(v)=2

Let , , ,… be the vertices of path . We label these vertices as follows:

1, 2 x 2, 2 x 2 , … , 2 x2

3, 3 x 2, 3 x 2 , . . ., 3 x2 ….. ….. ….. (2)

5, 5 x 2, 5 x2 , .. . , 5 x2

. . . .

where (2 − 1) 2 ≤ and ≥ 1, > 0. We observe that (2 − 1)2 divide (2 − 1) 2 ;(<)and (2 − 1) 2 does not divide (2m+1).

The remaining vertices are given other labels up to p. Then (n + 1) edges receives label 1 and (n + 1) edges receives label 2.

That is, (1) = (0) = (+ 1)

Therefore, | (0) − (1)| = 0

Subcase (ii) n is odd

We label as above case. The vertices of path is labeled as

1, 2 x 2, 2 x 2 , … , 2 x2

3, 3 x 2, 3 x 2 , . . . , 3 x 2 ….. ….. ….. (2) 5, 5 x 2, 5 x2 , .. . , 5 x2

. . . .

⌊ ⌉, ⌊ ⌉ x 2, ⌊_⌋ x2 , . . ⌊_⌋ x2

where (2 − 1) 2 ≤ and ≥ 1, > 0. We observe that (2 − 1)2 divide (2− 1) 2 ; (<) and (2 − 1) 2 does not divide (2m+1).

Then (1) = (0) = (+ 1)

Therefore, (0) − (1) = 0

Case(ii)

Let any of the vertex ;1 ≤ ≤ be duplicated.

Subcase (i) n is even

( ) = 2 ; ( ) = and = ; where and are prime numbers Let , , ,... be the vertices of path . We label these vertices as follows:

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1, 2 x 2, 2 x 2 , … , 2 x2
3, 3 x 2, 3 x 2 , . . . ,3 x 2 ….…..….. (2)
5, 5 x 2, 5 x2 , . . . ,5 x2 . . . .

Where (2 − 1) 2 ≤ and ≥ 1, > 0. We observe that (2 − 1)2 divide (2− 1) 2 ;(<) and (2 − 1) 2 does not divide (2 + 1). Remaining vertices are given other labels other than and

Subcase (ii) n is odd

We label as in the above case. The vertices of path are labeled upto ⌊ ⌋ , ⌊ ⌋ x 2, ⌊_⌋ x2 , . ., ⌊_⌋ x2
From the above cases, we observe that (1) = (0) = (+ 1)
Therefore, (0) − (1) = 0

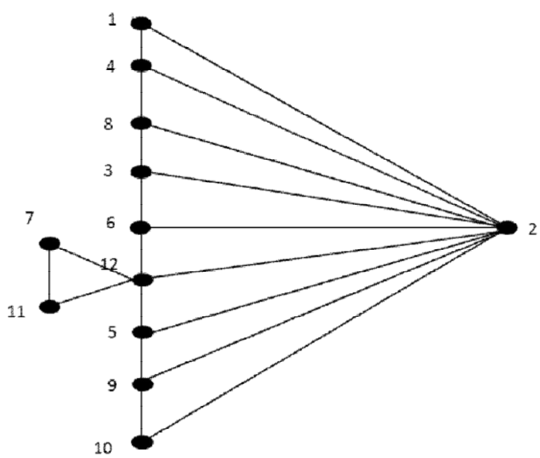

Figure: 6 Duplication of the vertex in

Case (iii) If the vertex is duplicated then the labeling as follows;
( ) = 2; ( ) = ; ( ) = 1; = Where and are prime numbers.
Other labels are given as in above cases. From the labeling, we observe that (1) = (0) = (+ 1)
Therefore, (0) − (1) = 0
From all cases (0) − (1) ≤ 1.
Hence duplication of a vertex by an edge is a divisor cordial graph.■

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