REVEALING THE LIMITING CHARACTERISTICS OF REGIONAL COMPOSITE QUANTILE REGRESSION WITHIN DIFFUSION MODELS

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Abstract: This paper introduces a novel approach for parameter estimation within the context of diffusion models. While composite quantile regression (CQR) has been applied effectively in classical linear regression models and more recently in general non-parametric regression models, its application in diffusion models has been limited. This research bridges this gap by extending CQR to estimate regression coefficients in diffusion models.

The diffusion model is considered within the framework of a filtered probability space $(\Omega, F, (Ft)t \ge 0, P)$, represented as: $dXt = \beta(t)b(Xt)dt + \sigma(Xt)dWt$, where $\beta(t)$ represents a time-dependent drift function, Wt is the standard Brownian motion, and $\beta(\vec{r})$ and $\beta(\vec{r})$ are known functions. Notably, Model (1.1) encompasses several well-known option pricing and interest rate term structure models, including Black and Scholes (1973), Vasicek (1977), Ho and Lee (1986), and Black, Derman, and Toy (1990), among others.

This study extends the applicability of CQR to diffusion models, offering a powerful tool for estimating regression coefficients in this context. It fills a significant research gap, providing a promising avenue for enhanced parameter estimation in the field of diffusion models.

Keywords: Composite quantile regression, parameter estimation, diffusion models, option pricing, interest rate term structure.

1. **Introduction**

Composite quantile regression (CQR) is proposed by Zou and Yuan (2008) for estimating regression coefficients
in classical linear regression models. More recently, Kai el.(2010) considers a general non-parametric regression
models by using CQR method. However, to our knowledge, little literature has researched parameter estimation
by CQR in diffusion models. This motivates us to consider estimating regression coefficients under the framework
of diffusion models. In this paper, we consider the diffusion model on a filtered probability space $(\Box, F, (Ft)t\Box 0, P)$
$(1.1) dX_t \square \square(t)b(X_t)dt \square \square(X_t)dW_t,$
$\Box(t)$ W is the standard Brownian motion. $b(\Box)$ and $\Box(\Box)$ are known
where is a time-dependent drift function and functions. Model (1.1) includes many famous option pricing models

 $\Box(t)$

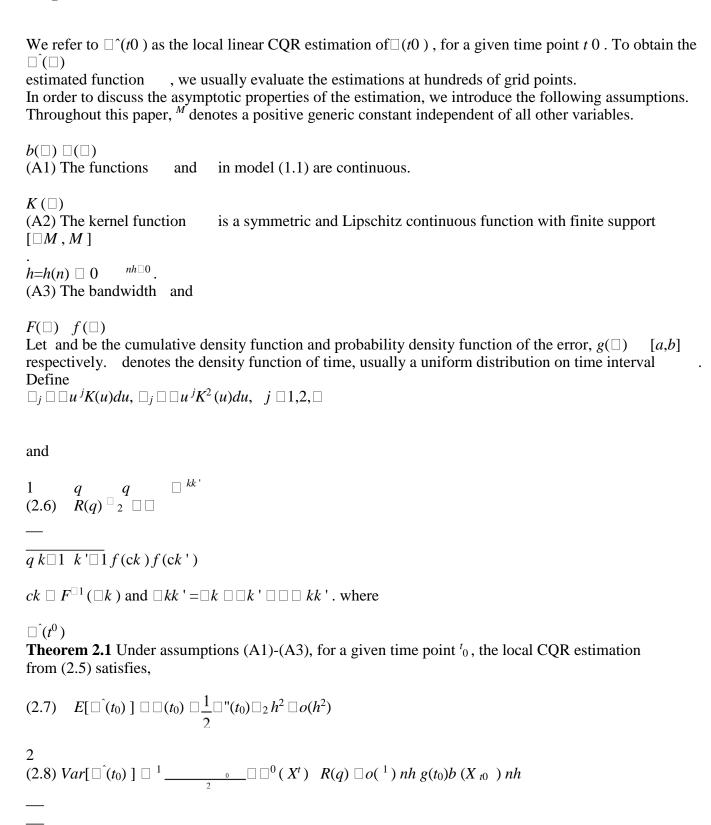
We allow being smooth in time. The techniques that we employ here are based on local linear fitting (see Fan and Gijbels(1996)) for the time-dependent parameter. The rest of this paper is organized as follows. In Section 2, we

and interest rate term structure models, such as Black and Scholes(1973), Vasicek(1977), Ho and Lee(1986),

Black, Derman and Toy (1990) and so on.

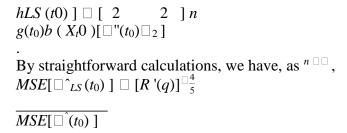
propose the local linear composite quantile regression estimation for the drift parameter and study its asymptotic properties. The asymptotic relative efficiency of the local estimation with respect to local least squares estimation is discussed in Section 3. The proof of result is given in Section 4.

2. Local estimation of the time-dependent parameter
$\{X^{ti}, i \square 1, 2, \square, n \square 1\} t^1 \square t^2 \square \square \square t^{n \square 1}$. Denote
Let the data be equally sampled at discrete time points,
$Yti \ \Box \ Xti \Box 1 \ \Box \ X \ ti$, $\Box ti \ \Box Wti \Box 1 \ \Box Wti$, and $\Box i \ \Box \ ti \Box 1 \ \Box ti$. Due to the independent increment property of
Brownian motion
$Wt, \Box ti$ are independent and normally distributed with mean zero and variance 1i. Thus, the discretized version of
the model (1.1) can be expressed as
$(2.1) Y_{ti} \square \square(t_i)b(X_{ti})\square_i \square \square(X_{ti})\square_i Z_{ti},$
Z^{ti} 1/ \Box i. The first-order discretized
where are independent and normally distributed with mean zero and variance approximation error to the
continuous-time model is extremely small according to the findings in Stanton (1997) and Fan and Zhang(2003),
this simplifies the estimation procedure.
Suppose the drift parameter $\Box(t)$ to be twice continuously differentiable in t . We can take $\Box(t)$ to be local
t^0 , we use the approximation linear fitting. That is, for a given time point
$(2.2) \square (t) \square \square (t_0) \square \square'(t_0)(t \square t_0)$
for in a small neighborhood of t0 . Let h denote the size of the neighborhood and $^{K(\square)}$ be a nonnegative weighted
function. h and $K(\square)$ are the bandwidth parameter and kernel function, respectively. Denoting $\square^{0} = \square(t0)$ and
$\Box^{1}\Box\Box'(t0)$, (2.2) can be expressed as
$(2.3) \square (t) \square \square_0 \square \square_1 (t \square t_0)$.
$\square(t)$
Now we propose the local linear CQR estimation of the drift parameter
. Let
k
$\Box k = \underline{\hspace{1cm}}$
$\square \square k$ (r) $\square \square kr \square \square I\{r \square 0\}, k \square 1, 2, \square, q$, which are q check loss functions at q quantile positions: $q \square 1$. Thus,
$\square(t)$
following the local CQR technique, can be estimated via minimizing the locally weighted CQR loss
$q n Yti \square 1$
$(2.4) \square \{\square \square_{\square k} \{ \qquad [b(X_{ti})] \square \square_{0k} \square \square_{1}(t_{i} \square t_{0}) \} K_{h}(t_{i} \square t_{0}) \}$
$ti \Box t \odot Kh$ ($ti \Box t0$)= K (where h and h is a properly selected bandwidth. Denote the minimizer of the locally
weighted
$(\Box^01,\Box^02,\Box,\Box^0q,\Box^1)T$
CQR loss (2.4) by . Then, we let
q
$q = (2.5) \ \Box\widehat{\ }(t_0) \ \Box^{\ 1} \ \Box\widehat{\ }_{0k}$
$\stackrel{-}{q}$ $k\Box 1$
1
1 k D1 i D1 i



and, as n	· • • • • • • • • • • • • • • • • • • •
2 (2.9) <i>n</i>	$h\{\Box \hat{t}(t)\Box \Box (t_0)\Box \Box \Box (t_0)\Box h^2\}\Box_L N(0,\phantom{AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA$
$\overline{2}$ g	$y(t_0)b(X_t0)$
\Box L mean where	ns convergence in distribution.
3. As We disculinear least \Box (t^0) .	Asymptotic relative efficiency uss the asymptotic relative efficiency(ARE) of the local linear CQR estimation with respect to the local ast squares estimation(see Fan and Gijbels(1996)) by comparing their mean-squared errors(MSE). From That is, theorem 2.1, we obtain the MSE
(3.1) M	$MSE[\Box \hat{\ }(t_0)\]\ \Box\ [^1\ \Box "(t_0)\Box_2]^2\ \Box\ ^1\ \underline{\qquad \qquad _0\ }\ \Box\ ^0\ (X^t)\ R(q)\ \Box o(h^4\ \Box\ ^1)$
$\frac{-}{2}$ n	
$\frac{1}{2}$ n	$hh g(t_0)b (X_{t0}) \qquad nh$
We obtain	in the optimal bandwidth via minimizing the MSE (3.1), denoted by
hopt (t ₀)] \square [\square
$g(t_0)b$ (X	$(X_t0)[\square''(t_0)\square_2]$
$\Box(t^0)$, d	lenoted by $\Box^{^{2}LS}(t^{0})$, is The MSE of the local linear least squares estimation of
2 (3.2) <i>N</i>	$MSE[\Box^{}_{LS}(t_0)] \Box [^1\Box''(t_0)\Box_2]^2h^4\Box^1\underline{\qquad}_2\underline{\qquad}\Box\Box^0\qquad (X^t) \Box o(h^4\Box^1)$
<u> </u>	
$\frac{-}{2}$ n	$hh g(t_0)b (X_{t0}) \qquad nh$
and the o	optimal bandwidth is
$\Box\Box$ 2(X opt) $1 \underline{1} \underline{0} \underline{t_0} \overline{5} \overline{b_5}$





Thus, the ARE of the local linear CQR estimation with respect to the local linear least squares estimation is $\Box \frac{4}{5}$

 $(3.3) \quad ARE(\Box (t_0), \Box LS(t_0)) \Box [R(q)]$

(3.3) reveals that the ARE depends only on the error distribution. The ARE we obtained is equal to that in Kai el.(2010).

 $ARE(\Box \hat{(}t^0),\Box^{\hat{L}S}(t^0))$ for some commonly seen error distributions. Table 1 in Kai Table 3.1 displays el.(2010) can be seen as ARE for more error distributions.

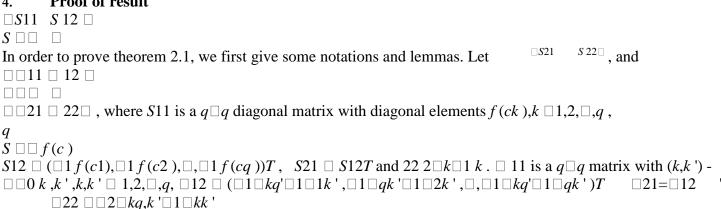
Table 3.1: Comparisons of $ARE(\Box^{\hat{}}(t0), \Box^{\hat{}}LS(t0))$ for the values of q

Error	$q \Box 1 q \Box 5$	$q \square 9 q \square 19$	<i>q</i> □ 99
N(0,1)	0.6968 0.9339	0.96590.9858	0.9980
Laplace	1.7411 1.2199	1.1548 1.0960	1.0296
$0.9N(0,1) \square 0.1N(0,10^2)$	4.0505 4.9128	4.70693.5444	1.1379

From Table 3.1, we can see that the local linear CQR estimation is more efficient than the local linear least squares estimation when the error distribution is not standard normal distribution. When the error distribution is N(0,1) and $q ext{ } ext$

CQR estimation performs well when the error conforms to the standard normal distribution too.

4. **Proof of result**



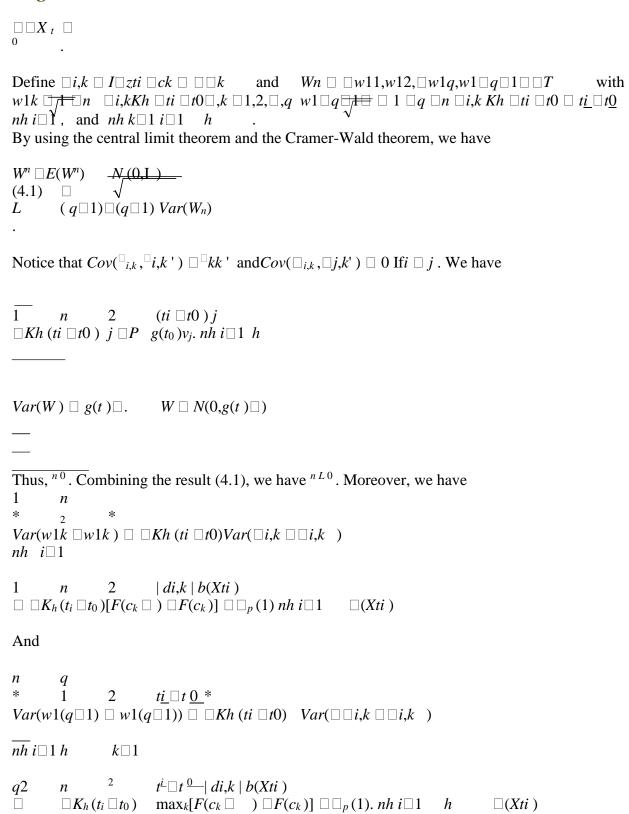
Original	Article												
element	, an	d.											
k Furthern	$(X) \square \square$	0k	0 $b(X_t0)$	k			\int_{1}^{1}	0			$\Box\Box(t)$		$_{t0}$ c \Box , v
$\begin{array}{ccc} & & & \\ & & v \\ \text{and} & & nh \end{array}$	$ \begin{array}{c c} \hline t0 \\ \hline \sqrt{} \\ i h \\ i (Xt0) \end{array} $	□ .W	$\Box i,k \Box$	$\Box uk$	$\Box ck \Box$ $\Box b(X)$ $\Box t0).$	Kti)	$\Box ri$ $b(Xt0)$		with r	i □ □((ti) 🗆 🗆	(t0) □[□' (t0)(ti
Define <i>i,k</i> 1(□ 1-		be \Box,q	<i>ti w</i> 1(□ ,2,		ŕ	titi q n □i		i □t0)		n	11	12	1q
Lemma 4 $q \square n \square i^*$ $L_n(\square) \square \square$	and $nh \ k \square$ J.1 Under as $k, k \ Kh(ti \square t)$ $k \square u_k \square \square \square$ $k \square 1 \ i \square$	ssumpti 0)□ v□□□	q	$n \square i^*$				uivalent q	to mini	mizing	g the foll	owing t	erm:
	$\square Z \square c \square d$ \square $\square W \square$ $\square (W_n^*)$			<i>T y</i>)	/								
$\Box = (u, u, 1)$ $1^{-1} q$	\Box , u , \Box) with respect	t to	, where	e									
$ \begin{array}{c} \square \\ Bn,k \square \square \\ I\square \square \ ti \ k \\ 0 \square \square Z \square \\ \end{array} $								k SSnn,,112			i ti □ z ,,1222 □		
	\square $\square X$ \square		<i>(</i>		$Z \square c \square$	$\Box\Box X$		□□□□,					
	$\Box X_t i \ \Box \ \Box \ \Box Kh \ \Box t i \ \Box$] <i>t</i> 0 □ □	$i \square \square S1$	11 with	$_{1}$ \square $_{0}$ i \square]1 $nh[$	$\square X t \square$	\Box , Sn,22	1 □SnT	,12 ,			
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 $h\sqrt{nh}$

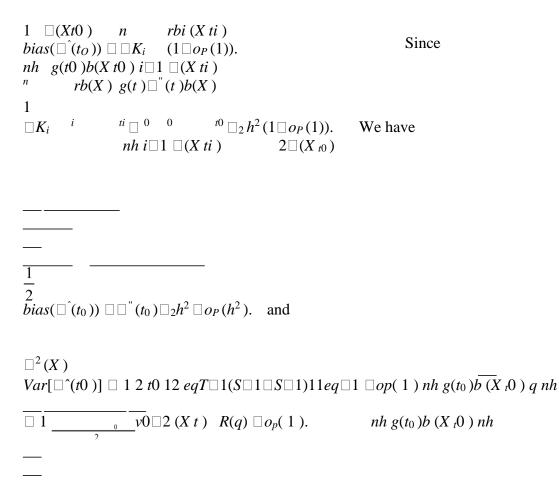
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 \sqrt{nh}

$Sn,12 \square \square \square \square n \ Kh \square ti \square t0 \square$	$t\underline{i} \Box t\underline{0} b \Box X \Box ti \Box i$	$i \square \square \square \square f \square c1 \square, f \square c2 \square, \square, f \square cq \square \square T$
	<i>t</i> □□ ,	
$ \Box q \qquad \Box ck \ \Box \Box n \ \Box \Box Kh \ \Box ti \ Sn,22 \ \Box \qquad f $	$\Box t0 \ \Box (ti \ \Box 2t0 \)2$	$b\Box X\Box ti\ \Box i\ \Box \Box\ \Box$
$ \begin{array}{c cccc} \hline \text{and} & \Box & h & nh \Box Xt \\ k \Box 1 & i \Box 1 \Box \end{array} $	□ □.	
The proof of lemma 4.1 is sim Proof of theorem 2.1 Using the results of Parzen(19)		d lemma 3 in Kai el.(2010).
$ \begin{array}{cccc} 1 & n & \Box ti \ \Box t0 \Box j \\ \Box Kh \ \Box ti \ \Box t0 \Box & j \ \Box P \ g \Box \\ $	$\Box t0\Box u j nh i\Box 1$	h
\square P means convergence in prowhere	bability. Thus,	
$g \Box t_0 \Box b \Box X_{t0} \Box$ $\Box \Box S_{11}$ $S_n \Box_P S \Box \Box$ $\Box \Box X t0 \Box \Box \Box X t0$	[$S_{12} \square$ \square S $22 \square$
According to lemma 4.1, we have L_{\square} \square	$\square^T S \square \square \square W_n^* \square^T \square$	$\Box \Box o_p \Box 1 \Box$ the convex function Since the convex function
L^{\square}	exity lemma in Poluniformly for . The	llard(1991), for any compact set, the quadratic us, we have
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\sqrt{n^*} \square o p \square 1 \square$	



Therefore,



This completes the proof.

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