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REVEALING THE LIMITING CHARACTERISTICS OF REGIONAL COMPOSITE QUANTILE REGRESSION WITHIN DIFFUSION MODELS

Li Wei

Department of Applied Mathematics, Fudan University, Shanghai, China

Abstract: This paper introduces a novel approach for parameter estimation within the context of diffusion models. While composite quantile regression (CQR) has been applied effectively in classical linear regression models and more recently in general non-parametric regression models, its application in diffusion models has been limited. This research bridges this gap by extending CQR to estimate regression coefficients in diffusion models.

The diffusion model is considered within the framework of a filtered probability space (Ω, F, (Ft)t≥0, P), represented as:

dXt = β(t)b(Xt)dt + σ(Xt)dWt, where β(t) represents a time-dependent drift function, Wt is the standard Brownian motion, and b(⃗) and σ(⃗) are known functions. Notably, Model (1.1) encompasses several well-known option pricing and interest rate term structure models, including Black and Scholes (1973), Vasicek (1977), Ho and Lee (1986), and Black, Derman, and Toy (1990), among others.

This study extends the applicability of CQR to diffusion models, offering a powerful tool for estimating regression coefficients in this context. It fills a significant research gap, providing a promising avenue for enhanced parameter estimation in the field of diffusion models.

Keywords: Composite quantile regression, parameter estimation, diffusion models, option pricing, interest rate term structure.

1. Introduction

Composite quantile regression (CQR) is proposed by Zou and Yuan (2008) for estimating regression coefficients in classical linear regression models. More recently, Kai el.(2010) considers a general non-parametric regression models by using CQR method. However, to our knowledge, little literature has researched parameter estimation by CQR in diffusion models. This motivates us to consider estimating regression coefficients under the framework of diffusion models. In this paper, we consider the diffusion model on a filtered probability space (□,F,(Ft)t□0,P)

(1.1) dX_t □□(t)b(X_t)dt □□(X_t)dW_t ,

□(t) W^t is the standard Brownian motion. b(□) and□(□) are known

where is a time-dependent drift function and functions. Model (1.1) includes many famous option pricing models and interest rate term structure models, such as Black and Scholes(1973), Vasicek(1977), Ho and Lee(1986), Black, Derman and Toy (1990) and so on.

□(t)

We allow being smooth in time. The techniques that we employ here are based on local linear fitting (see Fan and Gijbels(1996)) for the time-dependent parameter. The rest of this paper is organized as follows. In Section 2, we

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propose the local linear composite quantile regression estimation for the drift parameter and study its asymptotic properties. The asymptotic relative efficiency of the local estimation with respect to local least squares estimation is discussed in Section 3. The proof of result is given in Section 4.

2. Local estimation of the time-dependent parameter

{X ti ,i □ 1,2,□,n □1} t1 □ t2 □□□ tn□1. Denote

Let the data be equally sampled at discrete time points,

Yti □ Xti□1 □ X ti ,□ti □Wti□1 □Wti , and □i □ ti□1 □ti.Due to the independent increment property of Brownian motion

Wt ,□ti are independent and normally distributed with mean zero and variance[1i.]. Thus, the discretized version of the model (1.1) can be expressed as

(2.1) Yti □ □(ti)b(X ti)□i □□(X ti)□iZti ,

Zti 1/ □ i .The first-order discretized

where are independent and normally distributed with mean zero and variance approximation error to the continuous-time model is extremely small according to the findings in Stanton (1997) and Fan and Zhang(2003), this simplifies the estimation procedure.

Suppose the drift parameter □(t) to be twice continuously differentiable in t . We can take □(t) to be local

t 0 , we use the approximation linear fitting. That is, for a given time point

(2.2) □(t) □□(t0) □□'(t0)(t □t0)

for t in a small neighborhood of t0 . Let h denote the size of the neighborhood and K(□) be a nonnegative weighted function. h and K(□) are the bandwidth parameter and kernel function, respectively. Denoting □0 =□(t0) and

□1 □□'(t0) , (2.2) can be expressed as

(2.3) □(t) □□0 □□1(t □t0) .

□(t)

Now we propose the local linear CQR estimation of the drift parameter . Let

k

□k =____

□□k (r) □□kr □□I{r□0},k □ 1,2,□,q ,which are q check loss functions at q quantile positions: q □ 1 . Thus,

□(t)

following the local CQR technique, can be estimated via minimizing the locally weighted CQR loss

q n Y ti □1

(2.4) □ {□□□k { [b(Xti)] □□0k □□1(ti □t0)}Kh(ti □t0)}

ti □t 0) Kh (ti □t0)=K(where h and h is a properly selected bandwidth. Denote the minimizer of the locally weighted

(□ˆ01,□ˆ02,□,□ˆ0q,□ˆ1)T

CQR loss (2.4) by . Then, we let

q

(2.5) □ˆ(t0) □ 1 □□ˆ 0k

q k□1

---

1 k□1 i□1 i ,

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We refer to $\square\hat{}(t0)$ as the local linear CQR estimation of $\square(t0)$, for a given time point $t\ 0$. To obtain the
$\square\hat{}(\square)$
estimated function , we usually evaluate the estimations at hundreds of grid points.
In order to discuss the asymptotic properties of the estimation, we introduce the following assumptions. Throughout this paper, M denotes a positive generic constant independent of all other variables.

$b(\square)$ $\square(\square)$
(A1) The functions and in model (1.1) are continuous.

$K(\square)$
(A2) The kernel function is a symmetric and Lipschitz continuous function with finite support
$[\square M, M]$
.
$h=h(n) \square 0$ $nh\square 0$.
(A3) The bandwidth and

$F(\square)$ $f(\square)$
Let and be the cumulative density function and probability density function of the error, $g(\square)$ $[a,b]$ respectively. denotes the density function of time, usually a uniform distribution on time interval .
Define
$\square_j \square\square u^j K(u)du,\ \square_j \square\square u^j K^2(u)du,\ \ j \square 1,2,\square$

and

$1 \quad q \quad q \quad \square^{kk'}$
(2.6) $R(q) \square_2 \square\square$

$q\ k\square 1\ \ k'\square 1\ f(ck)\, f(ck')$

$ck \square F^{\square 1}(\square k)$ and $\square kk' = \square k \square\square k' \square\square\square kk'$. where

$\square\hat{}(t^0)$
Theorem 2.1 Under assumptions (A1)-(A3), for a given time point t_0, the local CQR estimation from (2.5) satisfies,

(2.7) $E[\square\hat{}(t_0)] \square\square(t_0) \square \frac{1}{2}\square''(t_0)\square_2 h^2 \square o(h^2)$

2
(2.8) $Var[\square\hat{}(t_0)] \square \frac{1}{\ \ \ _0\ \ \ }\square\square^0 (X^t)\ \ R(q) \square o(\frac{1}{nh})\ g(t_0)b(X_{t0})\ nh$

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and, as $n \to \infty$,

2

(2.9) $\sqrt{nh}\{\hat{\theta}(t_0) - \theta(t_0) - \frac{1}{2}\theta''(t_0)\mu_2 h^2\} \to_L N(0, \frac{\nu_0}{g(t_0)b(X_{t_0})}\sigma^2(X_t) R(q))$

where $\to_L$ means convergence in distribution.

3. Asymptotic relative efficiency

We discuss the asymptotic relative efficiency(ARE) of the local linear CQR estimation with respect to the local linear least squares estimation(see Fan and Gijbels(1996)) by comparing their mean-squared errors(MSE).From theorem 2.1, we obtain the MSE of $\hat{\theta}(t_0)$. That is,

(3.1) $MSE[\hat{\theta}(t_0)] = [\frac{1}{2}\theta''(t_0)\mu_2]^2 h^4 + \frac{1}{nh}\frac{\nu_0}{g(t_0)b(X_{t_0})}\sigma^2(X_t) R(q) + o(h^4 + \frac{1}{nh})$

We obtain the optimal bandwidth via minimizing the MSE (3.1), denoted by

$h_{opt}(t_0) = [\frac{\nu_0 \sigma^2(X_{t_0}) R(q)}{g(t_0)b(X_{t_0})[\theta''(t_0)\mu_2]^2}]^{\frac{1}{5}} n^{-\frac{1}{5}}$.

The MSE of the local linear least squares estimation of $\theta(t_0)$, denoted by $\hat{\theta}_{LS}(t_0)$, is

(3.2) $MSE[\hat{\theta}_{LS}(t_0)] = [\frac{1}{2}\theta''(t_0)\mu_2]^2 h^4 + \frac{1}{nh}\frac{\nu_0}{g(t_0)b(X_{t_0})}\sigma^2(X_t) + o(h^4 + \frac{1}{nh})$

and the optimal bandwidth is

$h_{opt} = [\frac{\nu_0 \sigma^2(X_{t_0})}{\ldots}]^{\frac{1}{5}} n^{-\frac{1}{5}}$

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$hLS(t0)$] □ [2 2] n
$g(t_0)b(X_t0)[\square''(t_0)\square_2]$
.
By straightforward calculations, we have, as ^{n} □□ ,
$MSE[\square\hat{}_{LS}(t_0)]$ □ $[R'(q)]^{\square\frac{4}{5}}$

$MSE[\square\hat{}(t_0)]$

Thus, the ARE of the local linear CQR estimation with respect to the local linear least squares estimation is
$\square\frac{4}{5}$
(3.3) $ARE(\square\hat{}(t_0),\square\hat{}_{LS}(t_0))$ □ $[R(q)]$

(3.3) reveals that the ARE depends only on the error distribution. The ARE we obtained is equal to that in Kai el.(2010).

$ARE(\square\hat{}(t^0),\square\hat{}^{LS}(t^0))$ for some commonly seen error distributions. Table 1 in Kai Table 3.1 displays el.(2010) can be seen as ARE for more error distributions.

Table 3.1: Comparisons of $ARE(\square\hat{}(t0),\square\hat{}LS(t0))$ **for the values of** q

Error	q □1 q □ 5	q □ 9 q □ 19	q □ 99
$N(0,1)$	0.6968 0.9339	0.96590.9858	0.9980
Laplace	1.7411 1.2199	1.1548 1.0960	1.0296
$0.9N(0,1)$ □$0.1N(0,10^2)$	4.0505 4.9128	4.70693.5444	1.1379

From Table 3.1, we can see that the local linear CQR estimation is more efficient than the local linear least squares estimation when the error distribution is not standard normal distribution. When the error distribution is $N(0,1)$ and q □ 1,5,9,19,99 , the $ARE(\square\hat{}(t0),\square\hat{}LS(t0))$ is very close to 1, which demonstrates that the local linear
CQR estimation performs well when the error conforms to the standard normal distribution too.

4. Proof of result

□$S11$ $S12$ □
S □□ □
In order to prove theorem 2.1, we first give some notations and lemmas. Let □$S21$ $S22$□, and
□□11 □ 12 □
□□□ □
□□21 □ 22□ , where $S11$ is a q□q diagonal matrix with diagonal elements $f(ck),k$ □1,2,□,q ,
q
S □□ $f(c)$
$S12$ □ (□1 $f(c1)$,□1 $f(c2)$,□,□1 $f(cq))T$, $S21$ □ $S12T$ and 22 2□k□1 k . □ 11 is a q□q matrix with (k,k') -
□□0 k,k',k,k' □ 1,2,□,q, □12 □ (□1□kq'□1□1k' ,□1□qk'□1□2k' ,□,□1□kq'□1□qk')T □21=□12 '
□22 □□2□kq,k'□1□kk'

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element , and .

□□ □(X) □□

k √0k 0 k √1 0 u □ nh □□ □□(t)□ t0 c □,v □ h nh□□ □□' (t) □

Furthermore, let □□ b(Xt0) □□ ,

1 □ ti □t0 □ di, k □i,k □ □uk □ ck □□ □□r i

□ v □√ □□b(Xti) b(Xt0) □□ with ri □ □(ti) □□(t0) □□' (t0)(ti □t0) .

and nh □ h □ .Write

□□ □(Xti) □(Xt0)□ □

Define i,k to be ti k i,k titi k . Let n 11 12 1q 1(q□1) with

□ 1 □n □ □,q w1(□ q□1) □ 1 □□q n □i□,kKh(ti □t0) ti □t 0

w1k □ □i,k Kh(ti □t0),k □1,2,

nh i□1 , and nh k□1 i□1 h .

Lemma 4.1 Under assumption (A1)-(A3), minimizing (2.4) is equivalent to minimizing the following term:

q □ n □i*,k Kh(ti □t0)□ q n □i*,kKh(ti □t0)(ti □t0) q

Ln(□) □ □uk □□□□v□□□□Bn,k (□)

k□1 □ i□1□ k□1 i□1k□1

□□ I □Z □c □d

□ □ □ □ □ T

b(X)□(X)□□ □ W □ (w ,w ,□,w ,w) /

□ 1/2 □T Sn□□ （Wn*）T □□ op (1)

□=(u ,u ,□,u ,□)

1 1 q with respect to , where

□

Bn,k □□□□□□i□n1 □□□□Kh □ti □t0 □□i,1 □□ □I□□ ti k d□ i,1b□Xti ti □ zb□Xttii □□□ -I□□ ti k di,1b□Xti ti □□□□□□□dz□ □□□ Sn □□□□SSnn,,1121 SSnn,,1222 □ □□□,

0 □ □Z □c □

□ □□ □X □ □□□X □□□□□□□□Z □c □ □□X □ □□□□□ □, □

□ n b□Xti □ □

Sn,11 □□□Kh □ti □t0 □ □ i □□S11 with □□ i□1 nh□ X t □□ , Sn,21 □SnT ,12 ,

1 □□X t □

√nh h√nh

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Sn,12 □□□□*n Kh* □*ti* □*t*0 □*ti*□*t*0 *b*□*X* □*ti* □*i* □□□□*f* □*c*1 □, *f* □*c*2 □,□, *f* □*cq* □□*T*

□□ *i*□1 *h* *nh*□ *X t* □□ ,

□*q* □*ck* □□*n* □□*Kh* □*ti* □*t*0 □(*ti* □2*t*0)2 *b*□*X*□*ti* □*i* □□ □
Sn,22 □ *f*

and □ *h* *nh*□ *Xt* □ □.
k□1 *i*□1 □

The proof of lemma 4.1 is similar to lemma 2 and lemma 3 in Kai el.(2010).

Proof of theorem 2.1

Using the results of Parzen(1962), we have

1 *n* □*ti* □*t*0□ *j*
□*Kh* □*ti* □*t*0□ *j* □*P g*□*t*0□*u j nh i*□1 *h*

□ *P* means convergence in probability. Thus,
where

g□t_0□*b*□X_{t0} □ *g*□t_0□*b*□X_{t0} S_{12} □
□□S_{11} □ *S*
S_n □$_P$ *S* □ □ 22□
□□*X t*0 □ □□*Xt*0 □ □*S*21 .

According to lemma 4.1, we have

L□□□□□ ^{1}*g*□*t*^{1} □ *b*□*X* ^{t0} □□^{T}*S*□□□W_n^* □^{T}□□ o_p □1□

L^n□□□□□□W^{n*}□^{T}□ converges in probability to the convex function Since the convex function

1 *g*□*t*0 □*b*□*X t*0 □ *T*
□ *S*□

2 □□*X* $_t$ □
0 , according to the convexity lemma in Pollard(1991), for any compact set, the quadratic
L□□□□ □
approximation to holds uniformly for . Thus, we have

ˆ □□ *g*□*t*0 □*b*□*X t*0 □ *S*□1*Wn** □*o p* □1□
□*n*

[1] .

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□□X_{t} □
0 .

Define □i,k □ I□zti □ck □ □□k and Wn □ □$w11,w12,$□$w1q,w1$□q□1□□T with
$w1k$ □1□n □i,kKh □ti □$t0$□,k □1,2,□,q $w1$□q□1□ □ 1 □q □n □i,k Kh □ti □$t0$ □ ti □$t0$
nh i□1 , and nh k□1 i□1 h .

By using the central limit theorem and the Cramer-Wald theorem, we have

W^n □$E(W^n)$ $N(0,I)$
(4.1) □
L (q□1)□(q□1) $Var(W_n)$
.

Notice that Cov(□$_{i,k}$,□i,k') □□kk' andCov(□$_{i,k}$,□j,k') □ 0 Ifi □ j . We have

1 n 2 (ti □$t0$) j
□Kh (ti □$t0$) j □P $g(t_0)v_j$. nh i□1 h

$Var(W)$ □ $g(t)$□. W □ $N(0,g(t)$□)

Thus, $^{n\,0}$. Combining the result (4.1), we have $^{n\,L\,0}$. Moreover, we have

1 n
* 2 *
$Var(w1k$ □$w1k)$ □ □Kh (ti □$t0$)Var(□i,k □□i,k)
nh i□1

1 n 2 | di,k | $b(Xti)$
□ □$K_h(t_i$ □$t_0)[F(c_k$ □) □$F(c_k)$] □□$_p$(1) nh i□1 □(Xti)

And

n q
* 1 2 ti □t 0 *
$Var(w1(q$□1) □ $w1(q$□1)) □ □Kh (ti □$t0$) Var(□□i,k □□i,k)
nh i□1 h k□1

$q2$ n 2 t^i□t ^{0}| di,k | $b(Xti)$
□ □$K_h(t_i$ □$t_0)$ max$_k$[$F(c_k$ □) □$F(c_k)$] □□$_p$(1). nh i□1 h □(Xti)

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$Var(w^{n*} \square w^{n}) \square\square ^{p} \quad ^{*}$

Therefore, (1) . Using Slutsky's theorem yields $w_n \square_L N(0, g(t_0)\square)$.

Thus,

$\square \square(Xt0) \quad \square 1 \quad * \quad \square 2 (Xt) \quad \square 1 \quad \square 1$

$\square_n \square S E(W_n) \square_L N(0, \frac{0}{2} S \square S) g(t_0) b(X_t) g(t_0) b(X_t)$

0 0

So the asymptotic bias of $\square\hat{}(tO)$ is:

$bias(\square\hat{}(tO)) \square 1 \square(Xt0) \quad \square q\ ck \square 1 \quad \square(Xt0) \quad eqT\square 1(S11)\square 1E(W1 * n)\ q\ b(Xt0)\ k\square 1\ q\ nh \quad g(t_0)b(X_{t0})$

$\square 1 \square(Xt0) \square^{q} ck \square 1 \square(Xt0) \quad \square^{n} Ki \square^{q} 1 \quad \square\square F(ck \square di,kb(Xt0)) \square F(ck)\square\square,\ q\ b(Xt0)\ k\square 1\ q\ nh\ g(t0)b(X$
$t0)\ i\square 1\ k\square 1\ f(ck) \square\sqrt{}\square \square(X\ ti) \quad \square\square$ where

$Ki \square Kh(ti \square t0), eq\square 1 \square (1,1,\square, 1)T$ and $W1*n \square(w11*, w12*, ...w1*q)T$.

q

c $\square 0$

z $\square k$, and

Note that ^{ti} is symmetric, thus $^{k\square 1}$

$_1 \quad ^{q} \quad 1 \square \quad d_{i,k}b(X_ti) \quad \square \quad rb_i(X_ti)$

$\square \square F(c_k \square \quad) \square F(c_k)\square\square\square \quad (1\square o_P(1)).\ q\ k\square 1\ f(ck) \square\square \quad \square(Xt) \quad \square\square \square(Xt)$

$i \quad i$

Therefore,

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1 □(*Xt*0) *n* *rbi* (*X ti*)
bias(□ˆ(*tO*)) □□*Ki* (1□*oP* (1)). Since
nh *g*(*t*0)*b*(*X t*0) *i*□1 □(*X ti*)
n *rb*(*X*) *g*(*t*)□" (*t*)*b*(*X*)

1
□*Ki* *i* *ti* □ 0 0 *t*0 □2 *h*2 (1□*oP* (1)). We have
nh i□1 □(*X ti*) 2□(*X t*0)

1
―
2
bias(□ˆ(*t*0)) □□" (*t*0)□2*h*2 □*oP* (*h*2). and

□2 (*X*)
Var[□ˆ(*t*0)] □ 1 2 *t*0 12 *eqT*□1(*S*□1□*S*□1)11*eq*□1 □*op*(1) *nh g*(*t*0)*b* (*X t*0) *q nh*

□ 1 0 *v*0□2 (*X t*) *R*(*q*) □*op*(1). *nh g*(*t*0)*b* (*X t*0) *nh*
2

This completes the proof.

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